Michael Moody

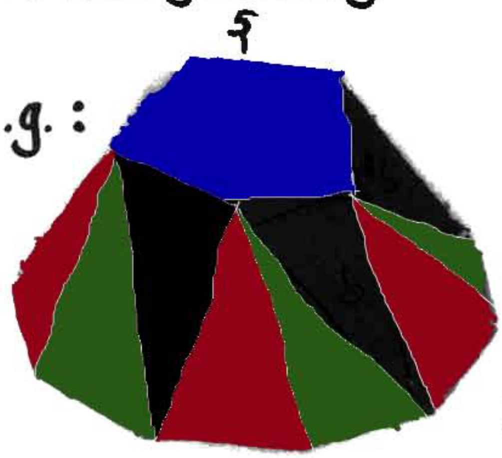
Intuitive Fourier Transforms

02/10/03

Possible types of rotational symmetry.

▷ Cyclic

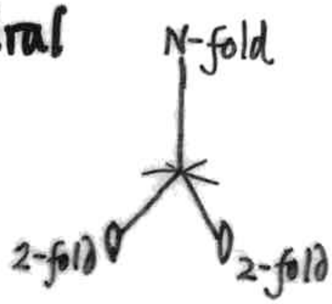
e.g.:



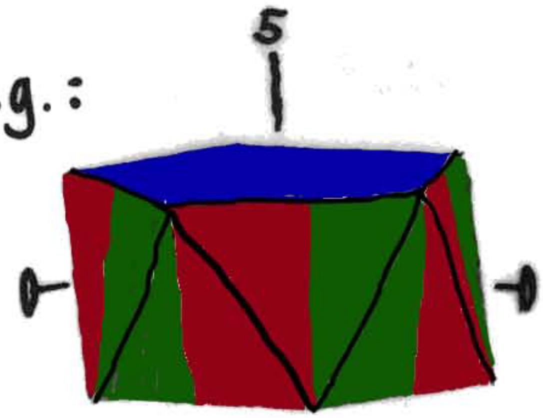
(Unchanged when turned through $\frac{1}{N}$th revolution)

N units

▷ Dihedral



e.g.:

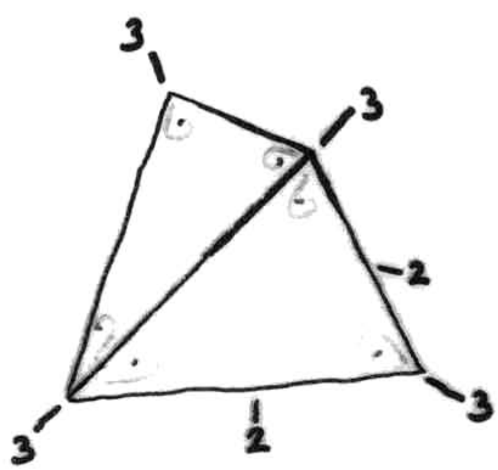


(Unchanged when turned through $\frac{1}{N}$th revolution OR upside-down)

2N units

▷ Tetrahedral

3 2-fold axes
4 3-fold axes

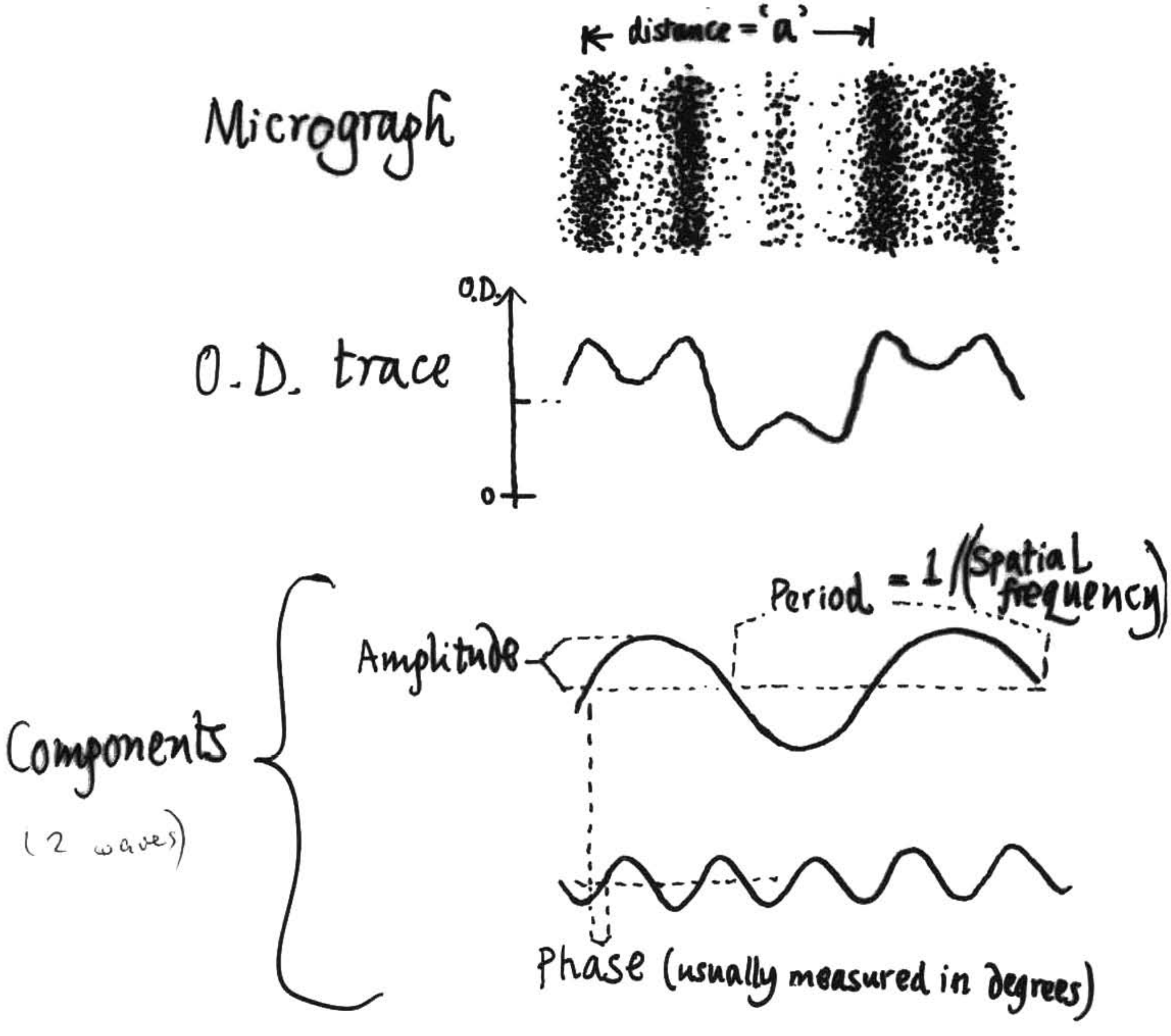


12 units

▷ Octahedral

▷ Icosahedral

1-D FOURIER TRANSFORMS



Fourier Analysis

curve → component ①, component ②, component ③, ...

Each component uses 3 numbers
- Period
- Amplitude
- Phase

Representation of these components.

— Point in 3-dimensional space ??
NO — since not practical for images of > 1 dimension

IDEALLY we would represent
1-D images as 1-D diagrammes
2-D images as 2-D diagrammes
... &c.

— Suitable representation
- one number on axis
- 2 remaining numbers as **vector**

<u>How should we do this?</u>

Examine the 3 numbers:

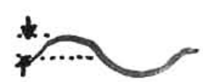
▷ Period measures <u>distance</u>

So appropriate for axis (1 distance axis in 1-D; 2 " " " 2-D)

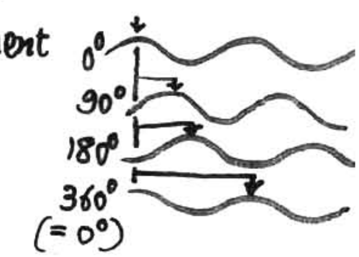
BUT period ranges from minimum → ∞
(~ resolution limit)

So use $\frac{1}{\text{period}}$ = <u>spatial frequency</u> (range 0 → max.)

▷ Amplitude ranges from 0 → maximum value
(no component)



▷ Phase measures
fractional shift of component
So use <u>degrees</u>
for phase 'angle'



Representation of components

Spatial frequency ($= \frac{1}{\text{period}}$) on axis

$\left\{\begin{array}{l}\text{Amplitude}\\ \text{Phase}\end{array}\right\}$ as **vector**

This gives the representation

component of <u>zero</u> spatial frequency, i.e. of ∞ period is uniform base level of O.D.

All vectors at multiples of $\frac{1}{a}$ (for repeating curve)

↳ So spatial frequency diagramme is called <u>reciprocal space</u>

Further extensions of this representation

▷ 2-D images need an extra axis.

▷ images that don't repeat exactly : vectors not only at multiples of $1/a$.

▷ images with amplitude & PHASE

(measure by O.D.)

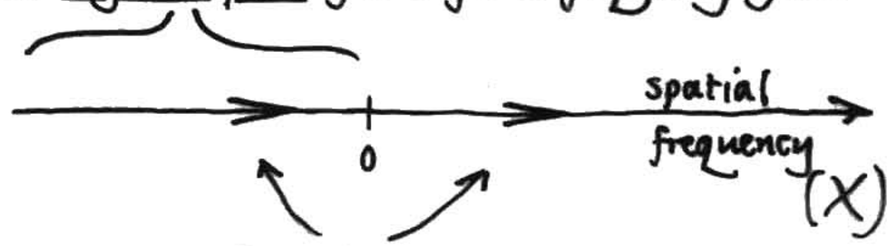
detect only by <u>interference</u> of wave + another wave

"real space place"

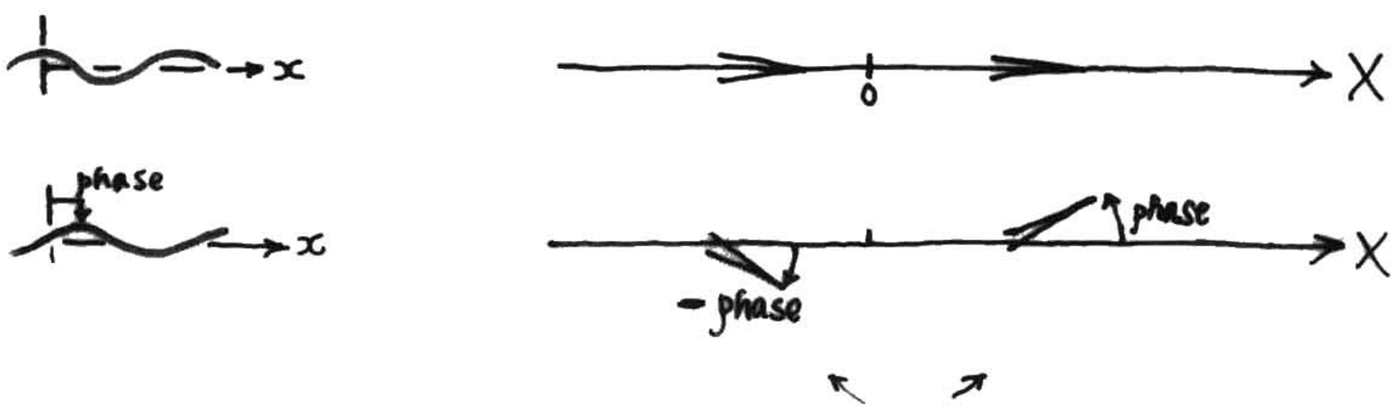
≠ reciprocal space phase

Representing the components of images with **phase** as well as **amplitude**:

Use the <u>negative part</u> of the spatial frequency axis.



2 vectors
represent a $\boxed{\text{pure amplitude wave}}$



"Friedel symmetry"

[Representation of pure phase wave postponed]

Questions:
1. why do we need phase representation for images?
2. why do we need negative frequencies?
Answer:
images not necessarily real

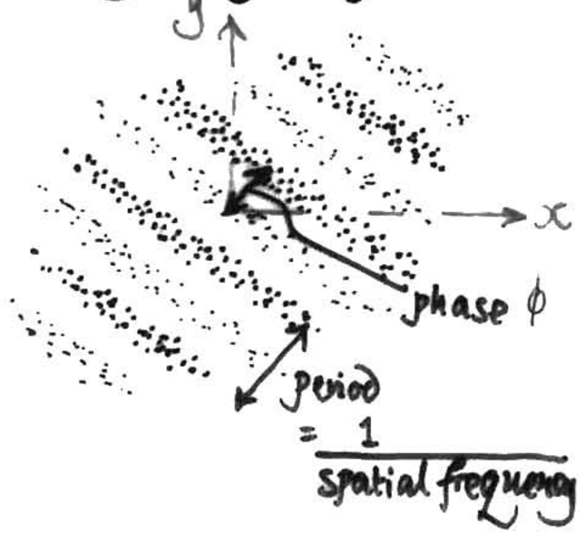
(not repeating)

What is the representation of an $\boxed{\text{ordinary}}$ picture?

An ordinary picture has very many components,

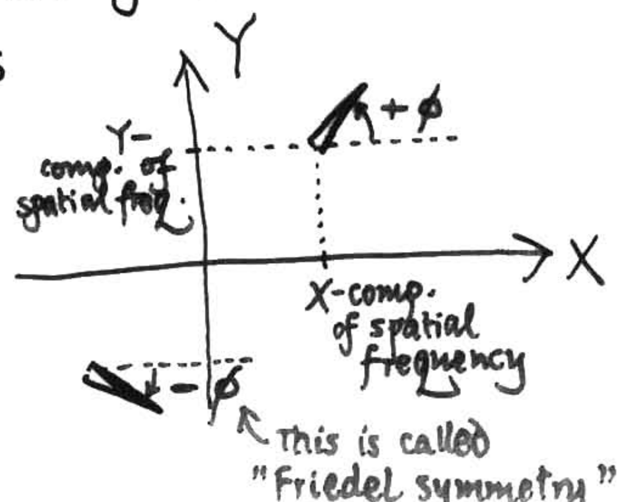
each with different

- spatial frequency
- phase
- amplitude



The representation of each component

is a pair of vectors



So

(not repeating)

the representation of an ordinary picture

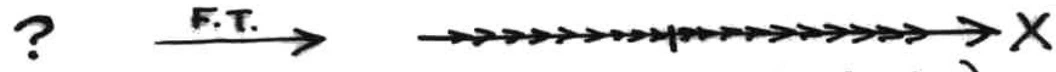
has <u>very</u> <u>many</u> <u>different</u> <u>pairs</u> of <u>vectors</u>

- even for a very simple picture.

"Fourier transform"

An example: F.T. of a <u>point</u>

Rationale for F.T. (vector) = phase-wave



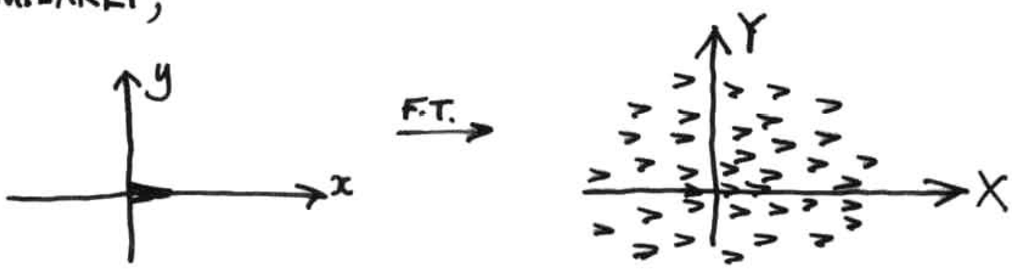
(uniform distribution of vectors)

Each vector pair must derive from a wave with <u>same phase</u> but <u>different period</u>

waves interfere here (destructively)

waves add here to give a <u>single peak</u>

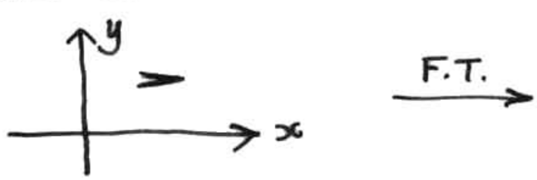
SIMILARLY,



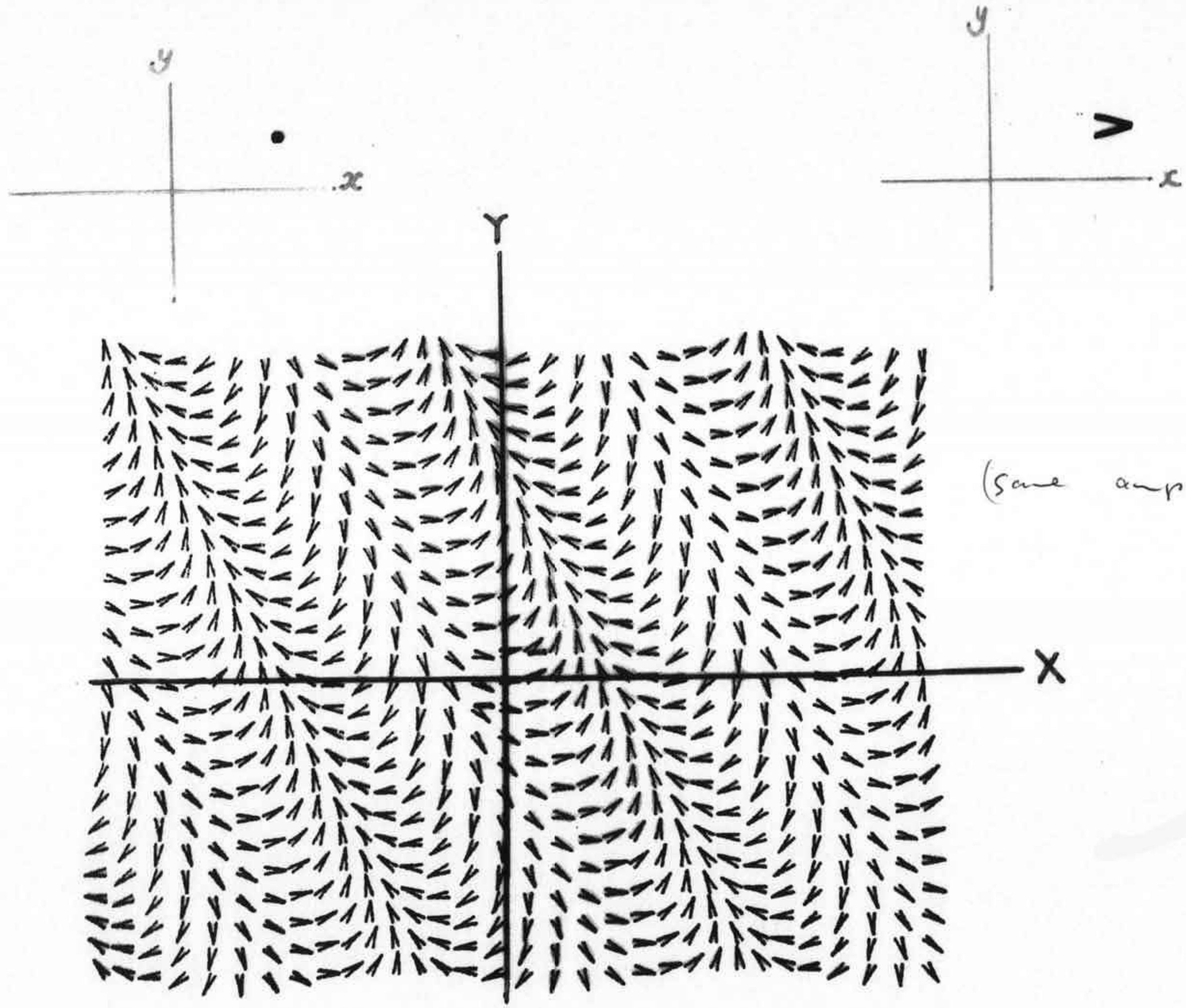
<u>BUT</u> translation (= uniform movement) of a curve changes PHASES of components, but not AMPLITUDES.

(so that you get the same set of sine-waves, just moved along to the new place, & adding together again.)

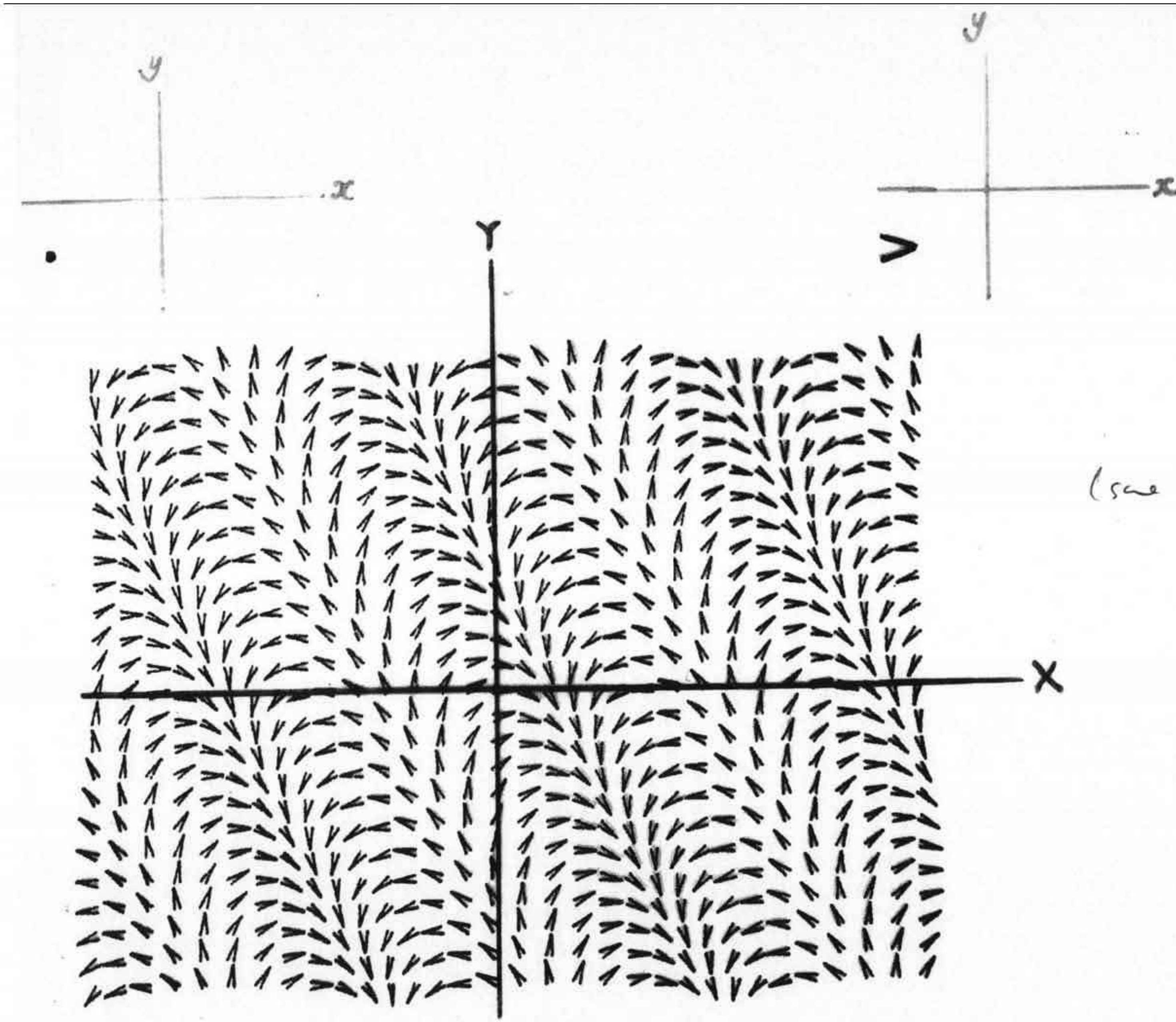
THEREFORE



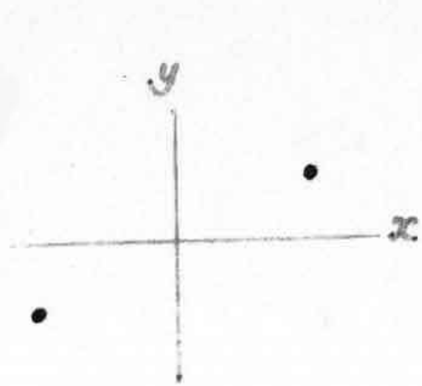
Phase wave (all vectors have same amplitude)

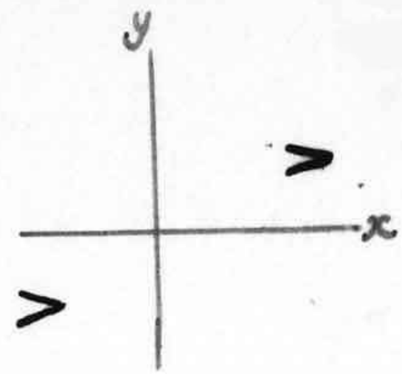


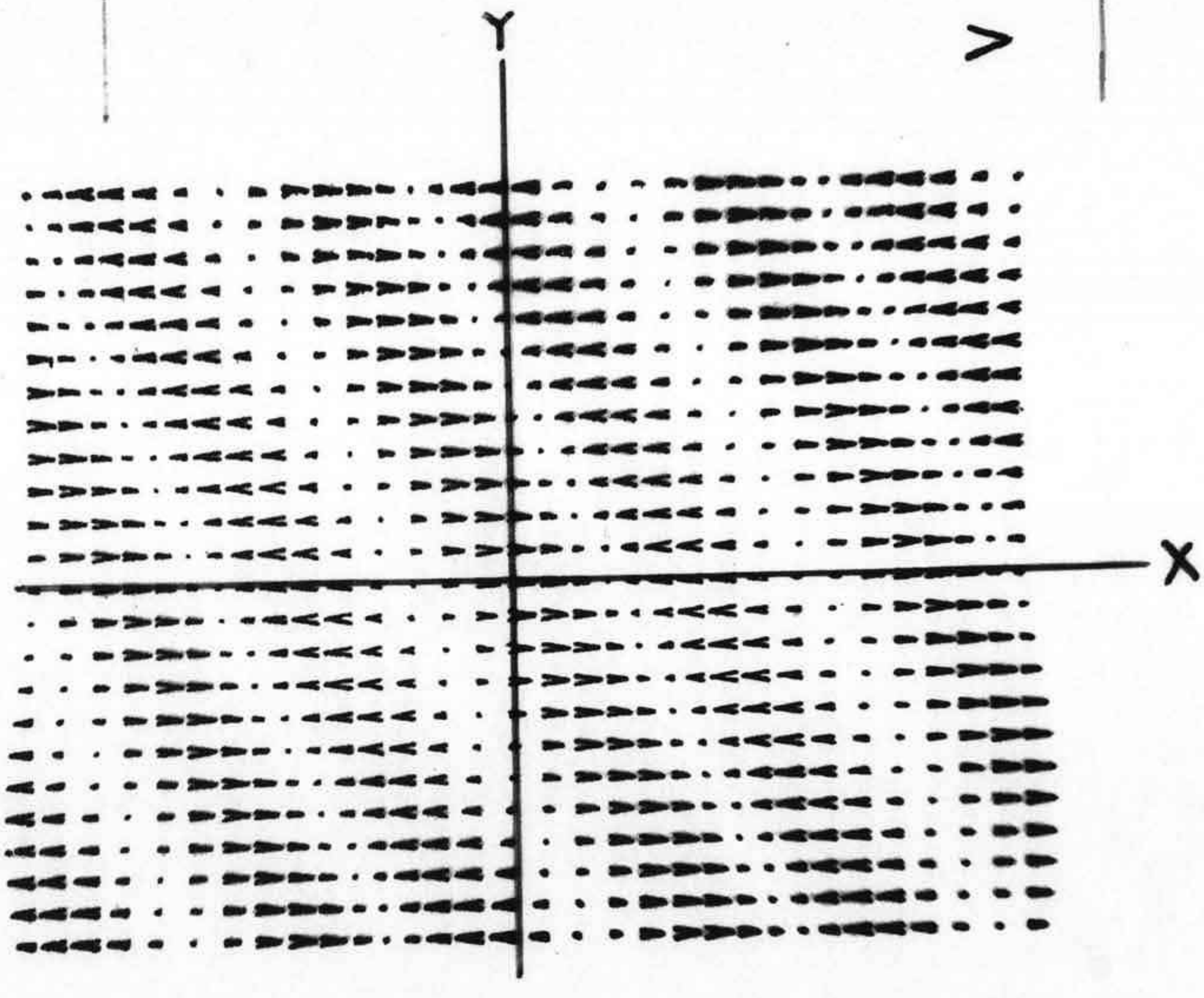
(same amplitude)



(same amplitude)

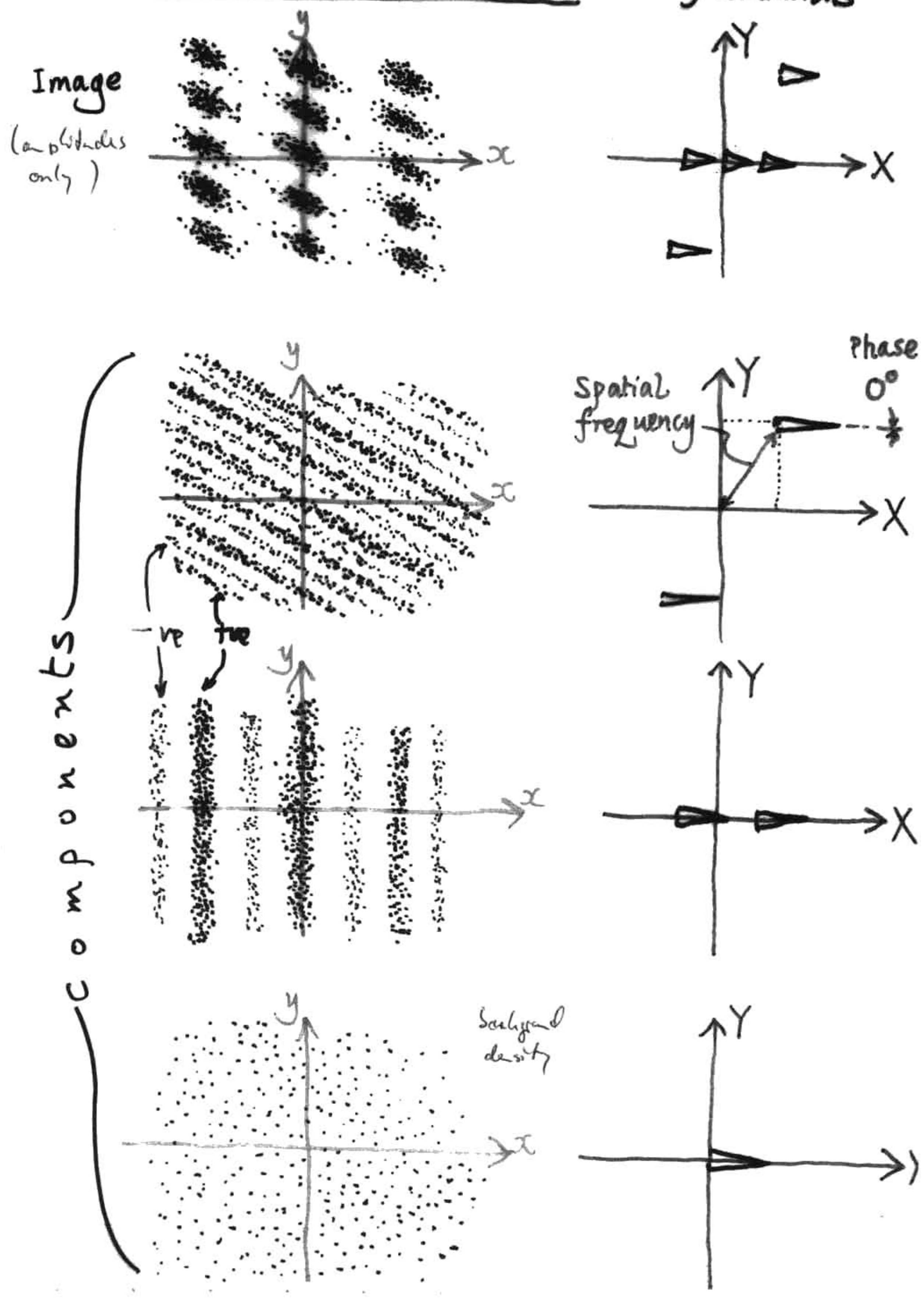






addition of
two diagrams

2-D FOURIER TRANSFORMS

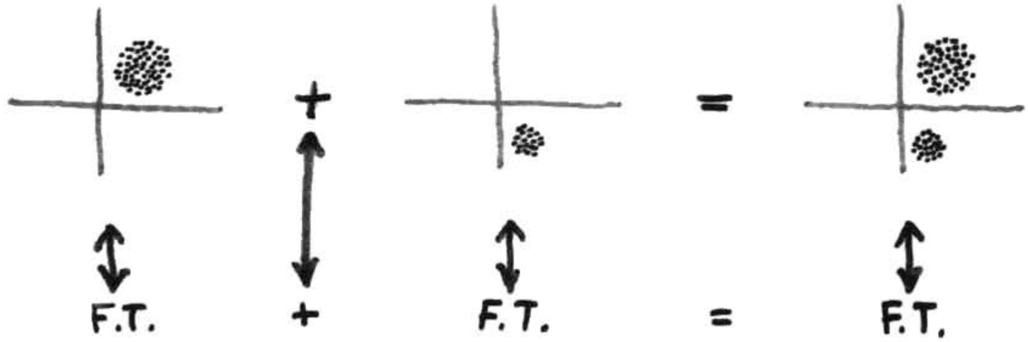


just 2 considered here

Six Basic Theorems concerning F.T.'s

Algebraic

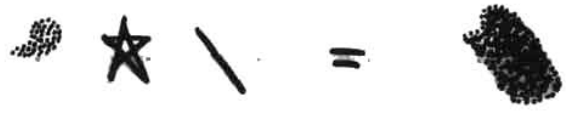
$\boxed{\pm \leftrightarrow \pm \quad \text{Linearity}}$



$\boxed{\times \leftrightarrow \star \quad \text{Convolution}}$ ← the **most important** rule

$\times$

$\star$ Each vector replaced by whole function $\times$ vector



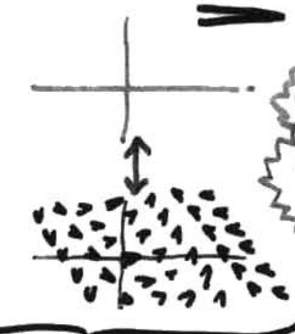
Isometric Movement

 $\leftrightarrow$ Rotation

(So F.T. has same point group symmetry as object.)

 $\leftrightarrow$ × complex wave Translation

Proof:

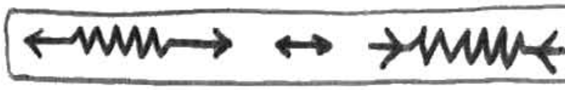
Translated object = object ★ 

↕ ↕ ↕ ↕

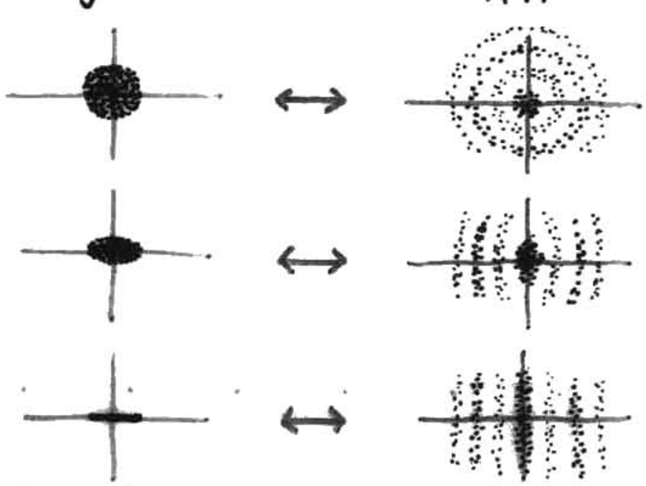
F.T. of translated object = F.T. of object ×

changes phases only, since vector lengths = 1

Distortion

 $\leftrightarrow$ Scale

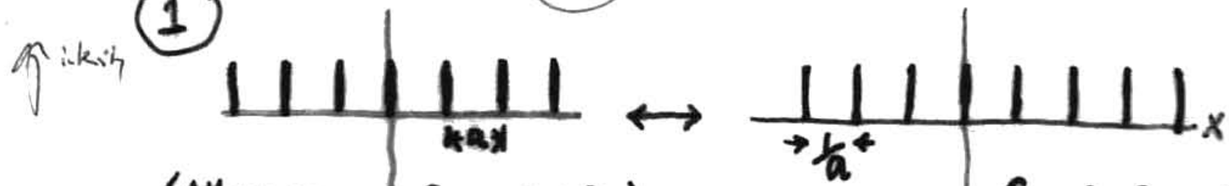
 $\leftrightarrow$ central section = projection plane (through origin) Projection



← section of F.T. corresponds to this

Examples of 1-D F.T.'s

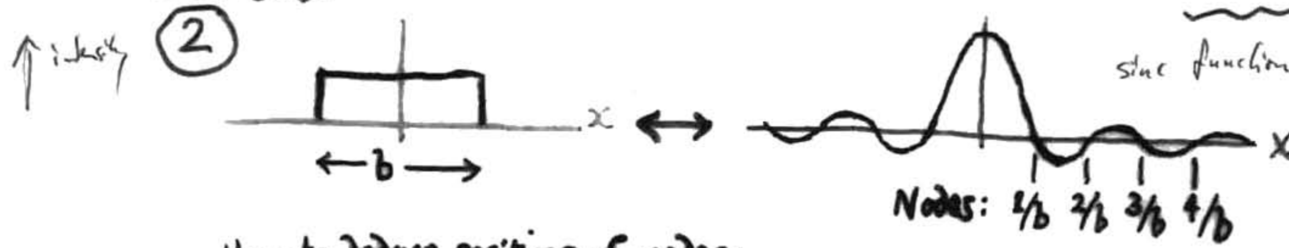
1



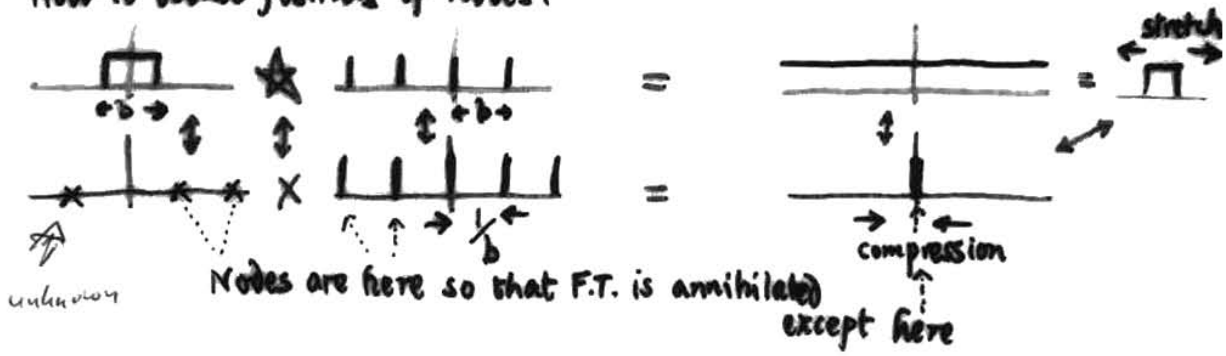
(All components have periods that are multiples of a)

(comb function)

2

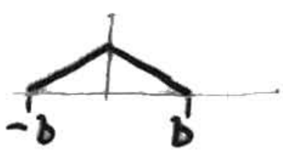


How to deduce positions of nodes:

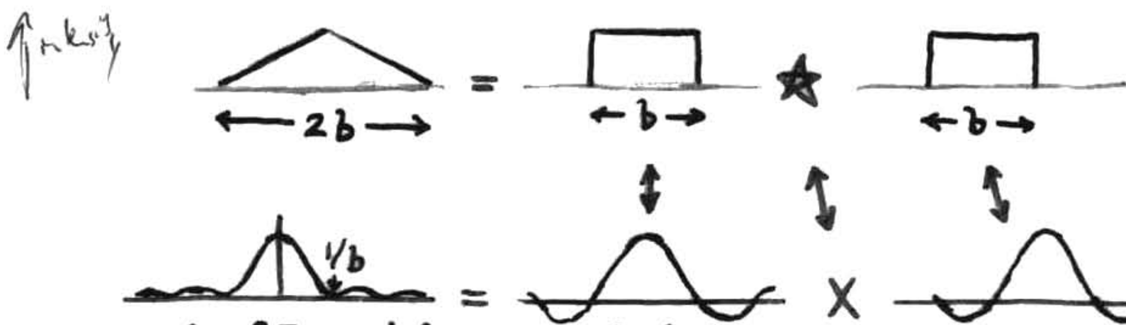


3

Finding F.T. of



Clue:



This F.T. must be square of

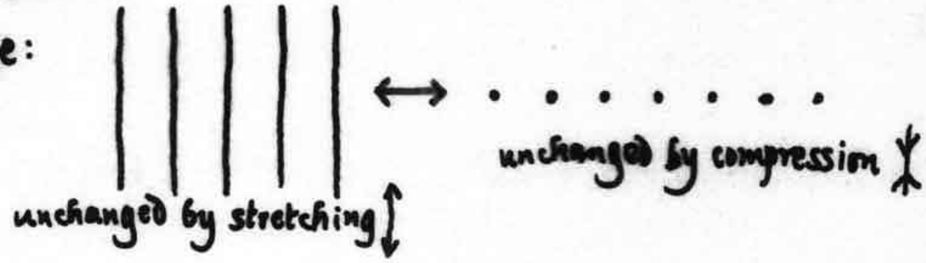
Hence removing sharp ends of rectangle weakens subsidiary ripples.

A further stage:

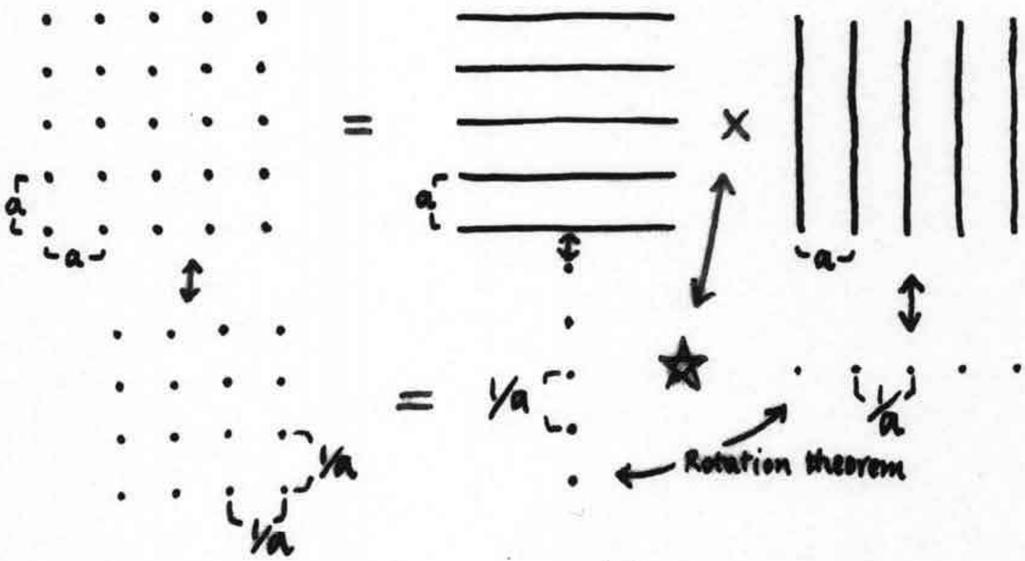
Examples of 2-D F.T.'s

① Square Lattice.

1st. stage:

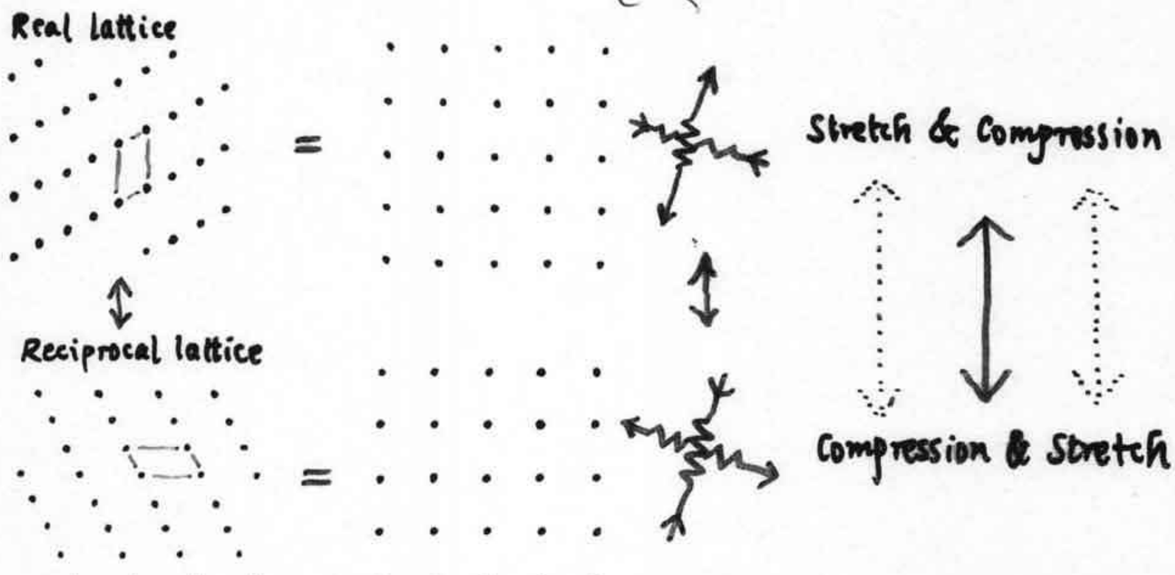


2nd. stage:



"applied to"

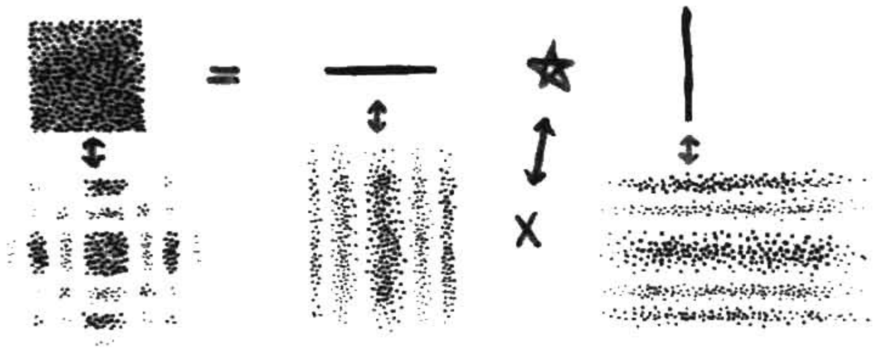
② Arbitrary Lattice



Shape of reciprocal lattice = shape of real lattice rotated by 90°

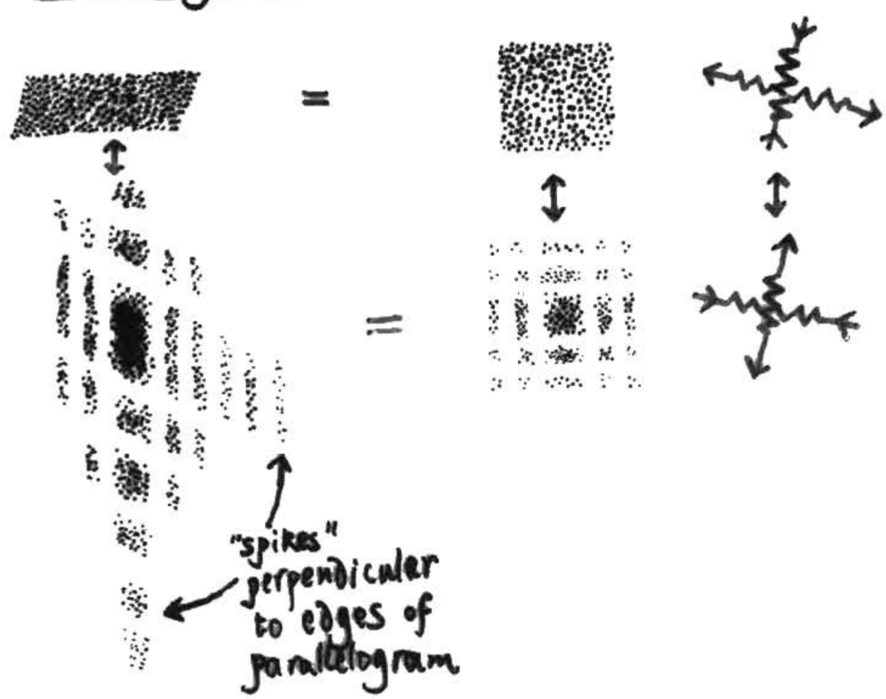
More examples of 2-D F.T.'s

③ Square



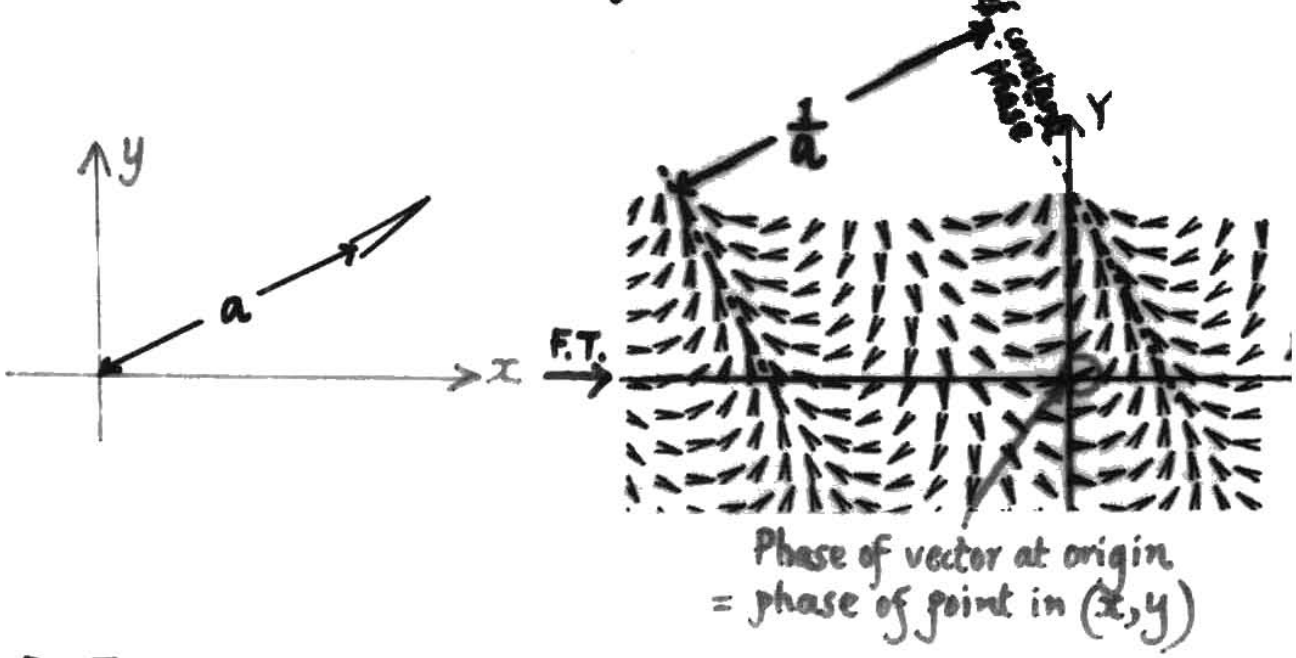
unknown

④ Parallelogram

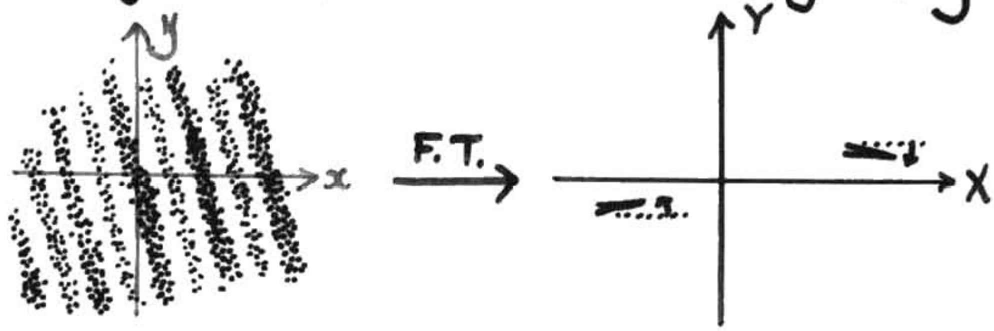


Summary

1. F.T. of one point is a complex wave



2. F.T. of a real structure has Friedel symmetry



3. F.T.[F.T.(structure)] = structure, rotated 180°.

Applies only to 2-D objects

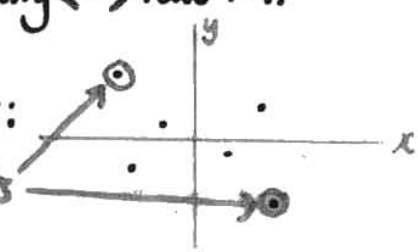
"Inversion theorem"; so write F.T. as $\longleftrightarrow$

Application of 2 and 3:

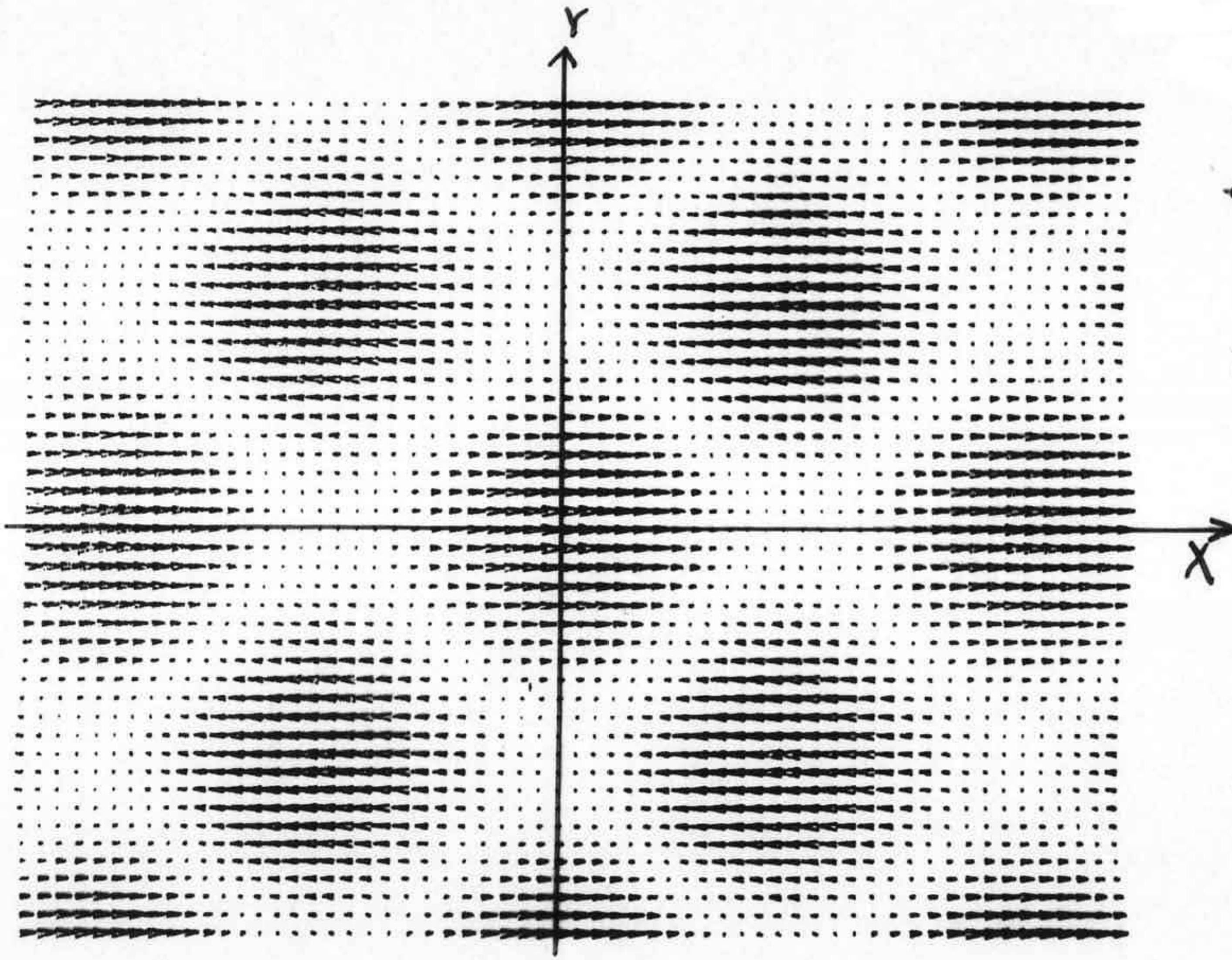
Structure with Friedel symmetry $\longleftrightarrow$ real F.T.

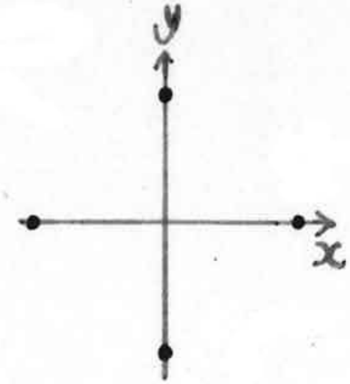
Real structure
$\updownarrow$
Structure with Friedel symmetry

But a **real** structure with Friedel symmetry is like this:

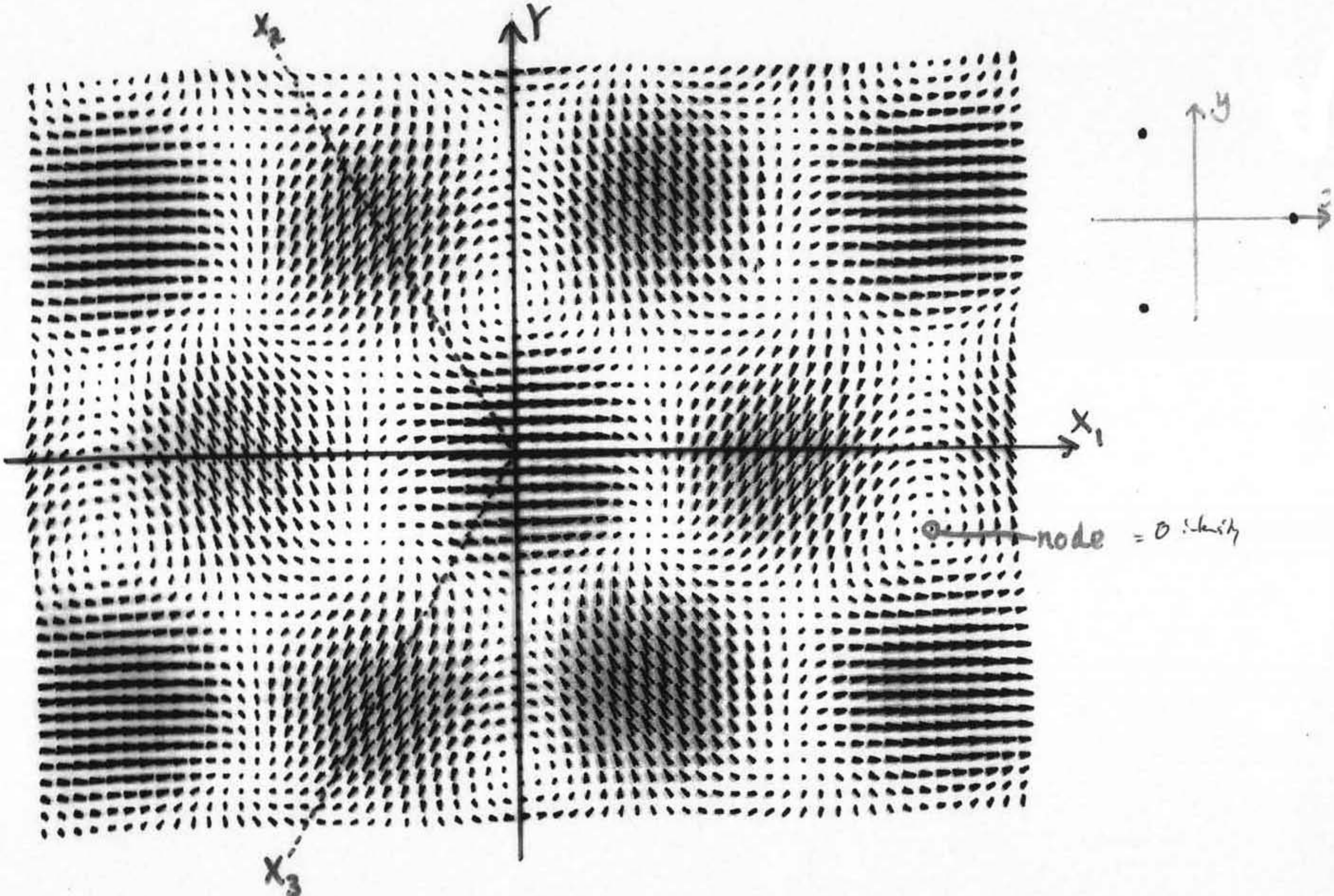


Symmetrical pairs of points

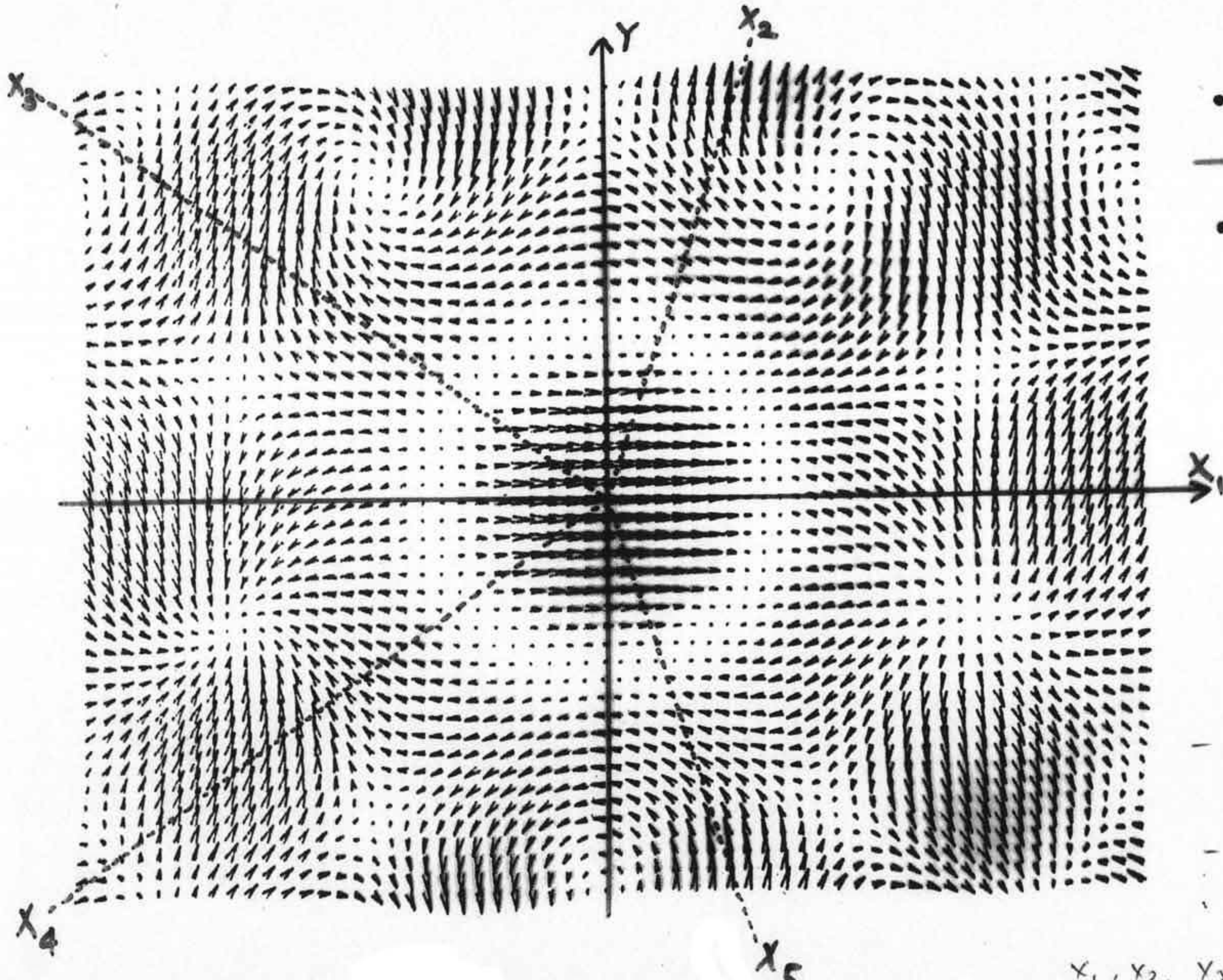


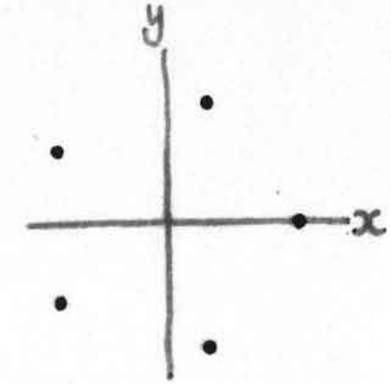


centrosymmetic,
so F.T. is real

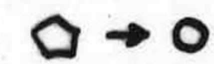


$\left.\begin{array}{l} x_1 \\ x_2 \\ x_3 \end{array}\right\}$ related by 120° rot.
see FT





Low resolution:



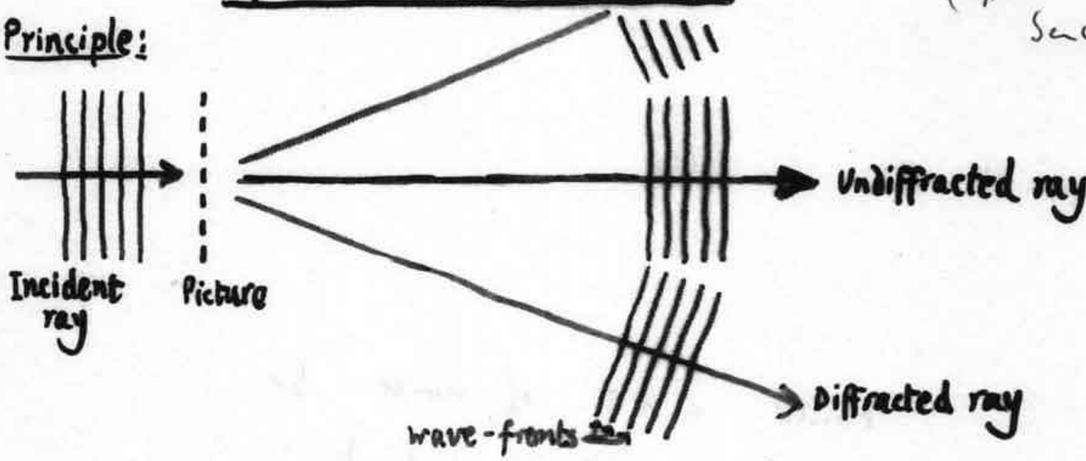
- central part ≈ FT of circle (at low resolution).
- FT of circle is real

X_1, X_2, X_3, X_4, X_5 see FT

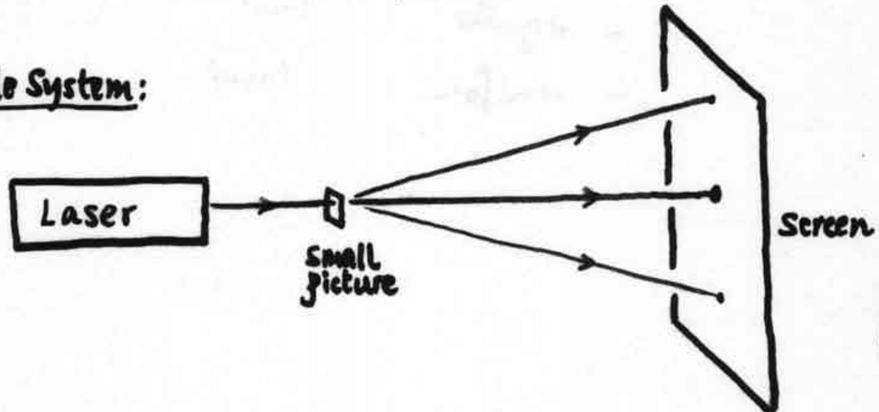
Optical Diffractometer

(optical Search)

Principle:

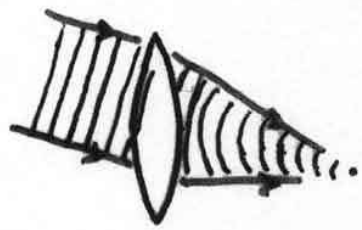


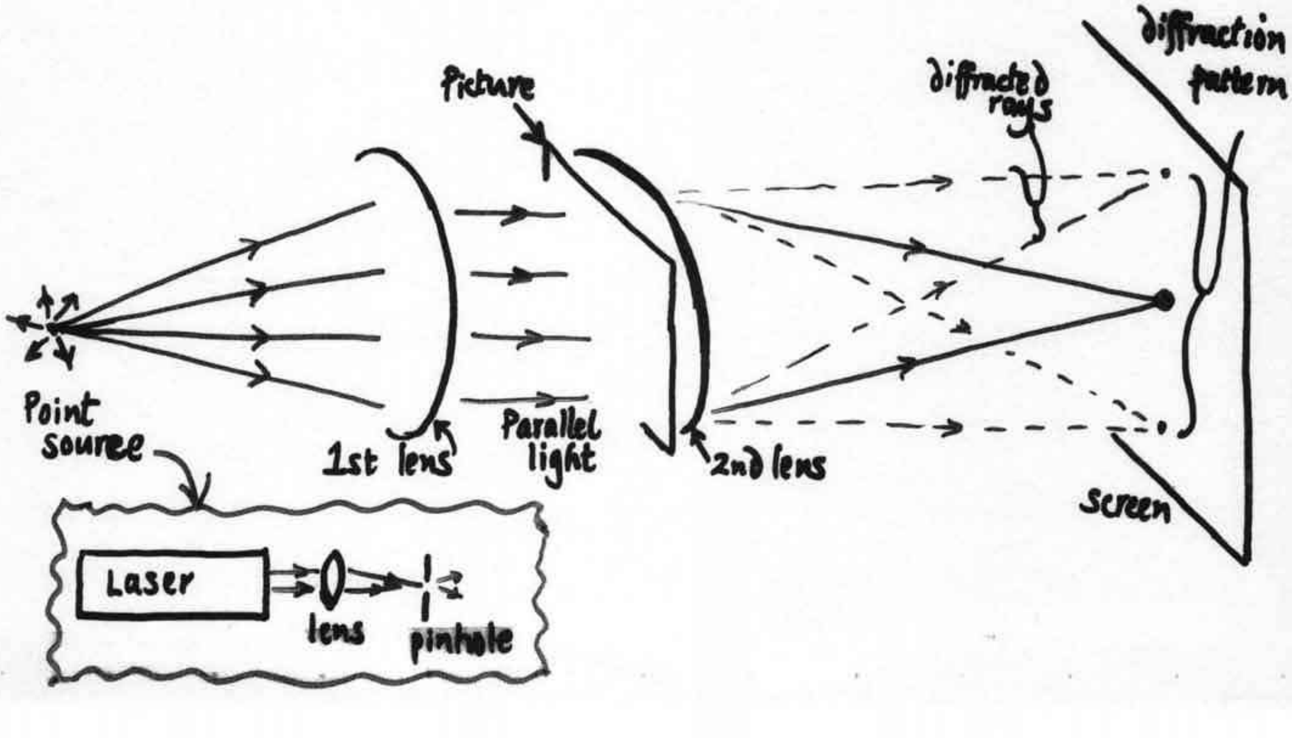
Simple System:



Diffraction pattern intensity proportional to (F.T. amplitude)^2

For **big pictures**, use lens:





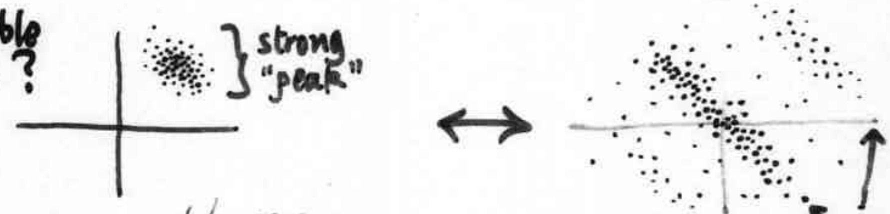
Value of Optical Diffraction Pattern

i.e., value of F.T. amplitude.

Example Noisy image of 2-D lattice

$\boxed{\text{What is F.T. (noise) ?}}$

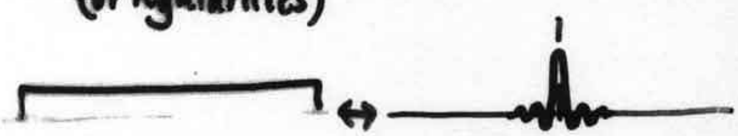
Possible F.T. ?



regularity; not noise

So ~~no~~ (can not have any) "peaks" or "concentrations" (or regularities) in F.T. (noise).

Also, not uniform.



uniform density ⟷ sharp peak at origin

Reconciliation:

$\boxed{\text{F.T.(noise) = noise}}$

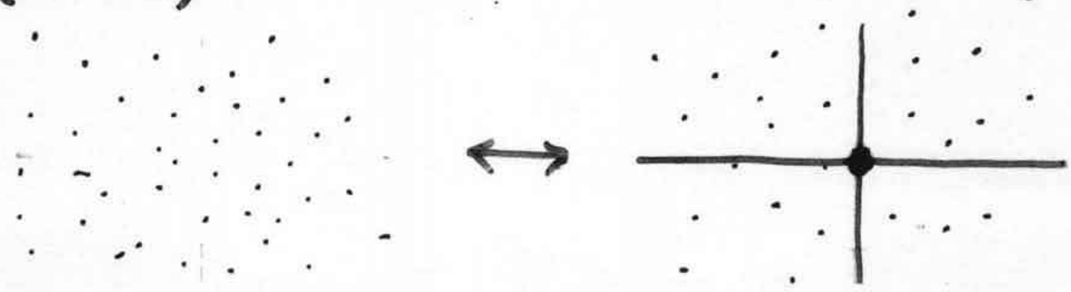
for +ve & −ve noise

Another way to look at this: noise is very complicated, so F.T. also very complicated.

When noise only +ve, add uniform density

+ve noise = ± ve noise + uniform density

F.T.(+ve noise) = (complex) noise + sharp peak at origin



F.T. of Noisy Image of 2-D Lattice

→ "Fog"

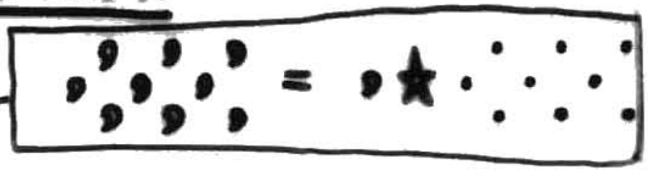


Image = unit cell $\star$ lattice + noise ← "fog"

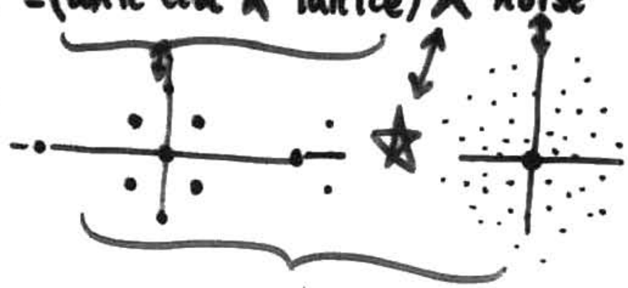
F.T.(Image) = (F.T. unit cell) X (reciprocal lattice) + (central peak + noise)

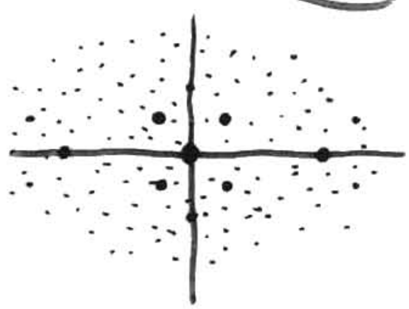
Strong spots show shape & size of lattice Despite noise

→ "Grain"

Image = (unit cell $\star$ lattice) X noise

F.T. (image) =

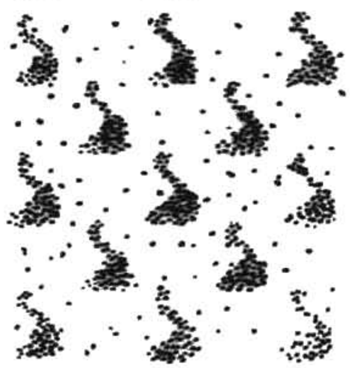




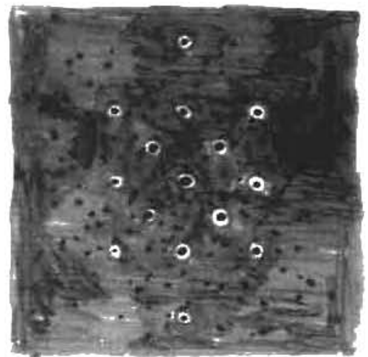
Measure lattice Despite noise

Examples of Optical Filtering

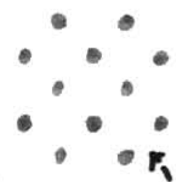
▶ **Lattice + noise**



$\longleftrightarrow$



$\longleftrightarrow$ filtered image



mask: (omits most noise)

filtered image $\longleftrightarrow$ F.T.(micrograph) $\times$ mask function

$$\boxed{\text{filtered image} = \text{micrograph} \star \text{F.T.}\left(\text{mask function}\right)}$$

E.g. if mask function = reciprocal lattice,

filtered image = micrograph $\star$ infinite lattice

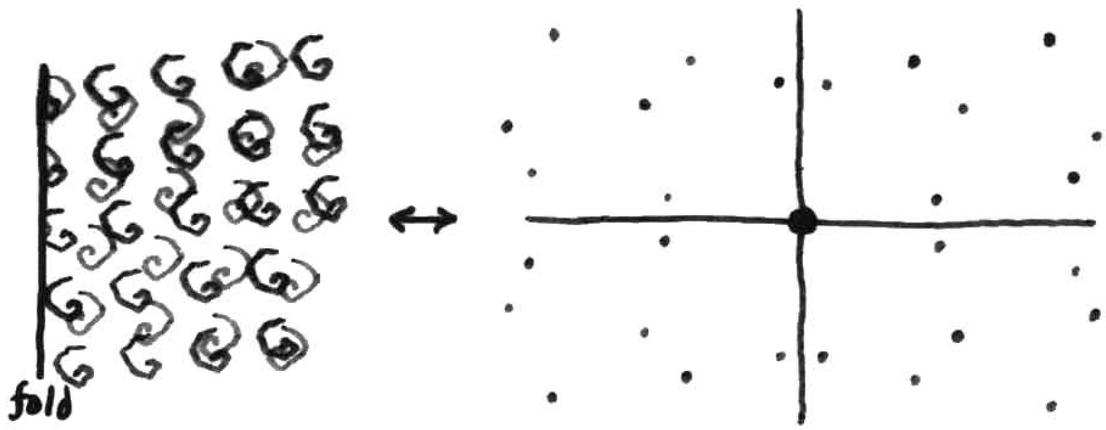
= averaged lattice of micrograph

But in practice, mask function $\neq$ infinite reciprocal lattice,

so optical filtering is not quite equivalent to (photographic) averaging.

Examples of Optical Filtering — contd

Folded Lattices



Filter out either • or • spots,
to obtain an image of one side.

These applications use binary filters
— 0% or 100% amplitude
— no phase change

General filters (variable amplitude & phase)
— various tricks with optical diffractometer (too difficult)
— use computer (<u>numerical filtering</u>)