THE UNIVERSITY *of* TEXAS
HEALTH SCIENCE CENTER AT HOUSTON
SCHOOL *of* HEALTH INFORMATION SCIENCES

Linear System Theory, Complex Numbers, Convolution

For students of HI 5323
"Image Processing"

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http://biomachina.org/courses/processing/06.html

Invariance

If some operation on a signal commutes with a particular transformation, that operation is invariant to that transformation:

$$U(Tf) = T(Uf)$$

The operation *U* is invariant to transformation *T*

Examples of Invariance

Operation

Transformation

	Distance	Direction	Intensity Difference
Translation	Yes	Yes	Yes
Rotation	Yes	No	Yes
Scaling	No	Yes	Yes
Warping	No	No	Yes
Uniform Amplification	Yes	Yes	No
Non-uniform Amp.	Yes	Yes	No

Linearity: Revisited

A function f is ***linear*** iff (if and only if):

$$f(ax + by) = af(x) + bf(y)$$

This can be broken down into two components

1. $f(ax) = af(x)$ (scalar multiplication)
2. $f(x + y) = f(x) + f(y)$ (addition)

Shift Invariance

Shift invariance: an operation is invariant to translation

Implication: shifting the input produces the same output with an equal shift

$$\text{if} \;\; x(t) \rightarrow y(t)$$

$$\text{then} \;\; x(t+T) \rightarrow y(t+T)$$

Systems

Linearity and shift invariance are nice properties for a signal-processing operation to have

- Input devices
- Output devices
- Processing

In signal processing, a transformation that is linear and shift invariant is called a ***system.***

Reality Check

No physical device is really a system:

- Linearity is limited by saturation
- Shift invariance is limited by sampling duration or field of view
- Random noise isn't linear

Impulses

One way of probing what a system does is to test it on a single input point (a single spike in the signal, a single point of light, etc.)

Mathematically, a perfect single-point input is written as:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

This is called the Dirac *delta function*

Impulses (cont.)

Multiplying a delta function by a constant multiplies the integrated area:

$$\int_{-\infty}^{\infty} a\ \delta(t)dt = a$$

Impulse Response

Because a system is shift-invariant, it responds the same everywhere:

$$\delta(t) \rightarrow h(t)$$

implies

$$\delta(t + T) \rightarrow h(t + T)$$

This response $h(t)$ is called the *impulse response* or *point spread function*

Impulse Response

Because a system is linear, the response to a multiplied impulse is the same as the multiple times the response:

$$\delta(t) \rightarrow h(t)$$

implies

$$a\,\delta(t) \rightarrow a\,h(t)$$

Impulse Response

Because a system is linear, the response to two impulses is the same as the sum of the two responses individually:

$$\delta(t) \rightarrow h(t)$$

$$\delta(t + T) \rightarrow h(t + T)$$

Implies

$$\delta(t) + \delta(t + T) \rightarrow h(t) + h(t + T)$$

Impulse Response

Putting it all together:

$$\delta(t) \rightarrow h(t)$$

implies

$$a\,\delta(t) + b\,\delta(t + T) \rightarrow a\,h(t) + b\,h(t + T)$$

Implication: If you know the impulse response at any point, you know everything there is to know about the system!

Complex Numbers: Review

A complex number is one of the form:

$$a + bi$$

where

$$i = \sqrt{-1}$$

a: real part

b: imaginary part

Complex Arithmetic

When you add two complex numbers, the real and imaginary parts add independently:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

When you multiply two complex numbers, you cross-multiply them like you would polynomials:

$$\begin{aligned}(a + bi) \times (c + di) &= ac + a(di) + (bi)c + (bi)(di) \\ &= ac + (ad + bc)i + (bd)(i^2) \\ &= ac + (ad + bc)i - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Polynomial Multiplication

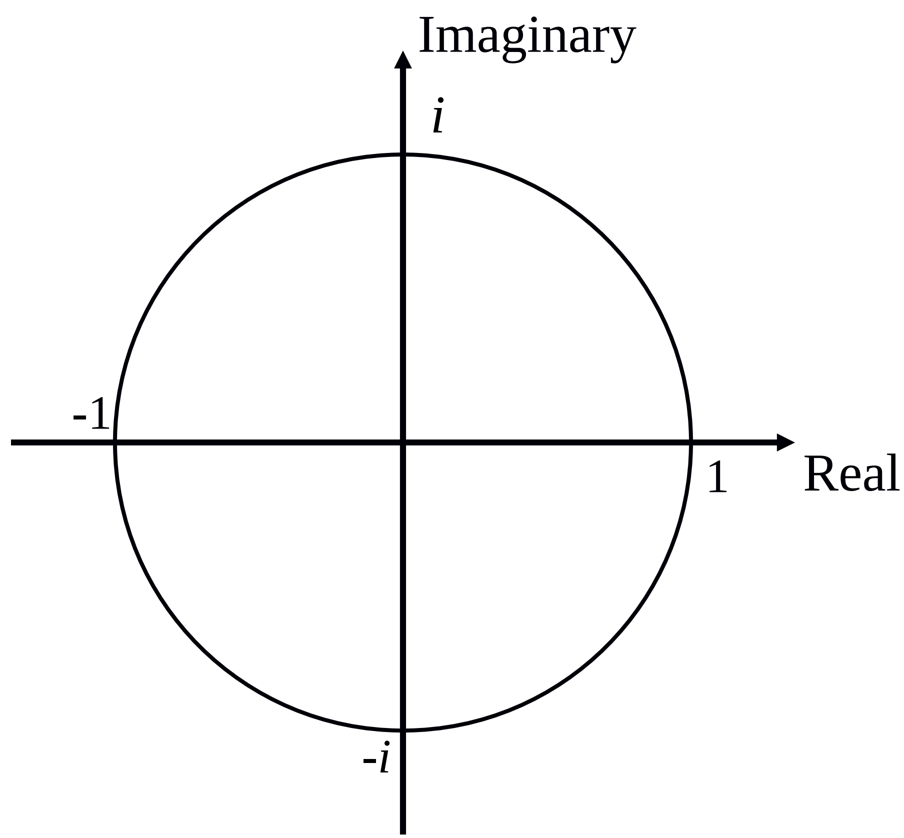
$$p_1(x) \;=\; 3\,x^2 + 2\,x + 4$$

$$p_2(x) \;=\; 2\,x^2 + 5\,x + 1$$

$$p_1(x)\,p_2(x) \;=\; \_\_\_\_\_x^4 + \_\_\_\_\_x^3 + \_\_\_\_\_x^2 + \_\_\_\_\_x + \_\_\_\_\_$$

The Complex Plane

Complex numbers can be thought of as vectors in the complex plane with basis vectors (1, 0) and (0, *i*):



Magnitude and Phase

The length of a complex number is its *magnitude*:

$$|a + bi| = \sqrt{a^2 + b^2}$$

The angle from the real-number axis is its *phase*:

$$\phi\,(a + bi) = \tan^{-1}(b / a)$$

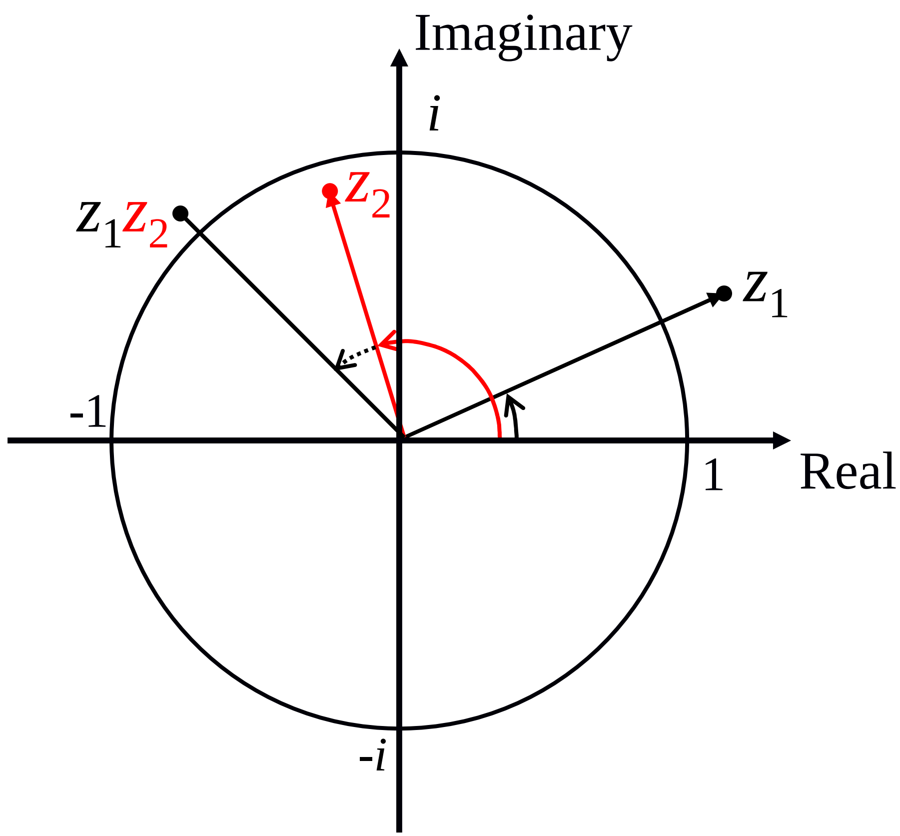
When you multiply two complex numbers, their magnitudes multiply

$$|z_1 z_2| = |z_1||z_2|$$

And their phases add

$$\phi\,(z_1 z_2) = \phi\,(z_1) + \phi\,(z_1)$$

The Complex Plane: Magnitude and Phase



Complex Conjugates

If $z = a + bi$ is a complex number, then its complex conjugate is:

$$z^* = a - bi$$

The complex conjugate z^* has the same magnitude but opposite phase

When you add z to z^*, the imaginary parts cancel and you get a real number:

$$(a + bi) + (a - bi) = 2a$$

When you multiply z to z^*, you get the real number equal to $|z|^2$:

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$$

Complex Division

If $z_1 = a + bi$, $z_2 = c + di$, $z = z_1 / z_2$,

the division can be accomplished by multiplying the numerator and denominator by the complex conjugate of the denominator:

$$z = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \left(\frac{ac+bd}{c^2+d^2} \right) + i \left(\frac{bc-ad}{c^2+d^2} \right)$$

Euler's Formula

- Remember that under complex multiplication:
  - Magnitudes multiply
  - Phases add
- Under what other quantity/operation does multiplication result in an addition?
  - Exponentiation: $c^a c^b = c^{a+b}$ (for some constant c)
- If we have two numbers of the form $m \cdot c^a$ (where c is some constant), then multiplying we get:

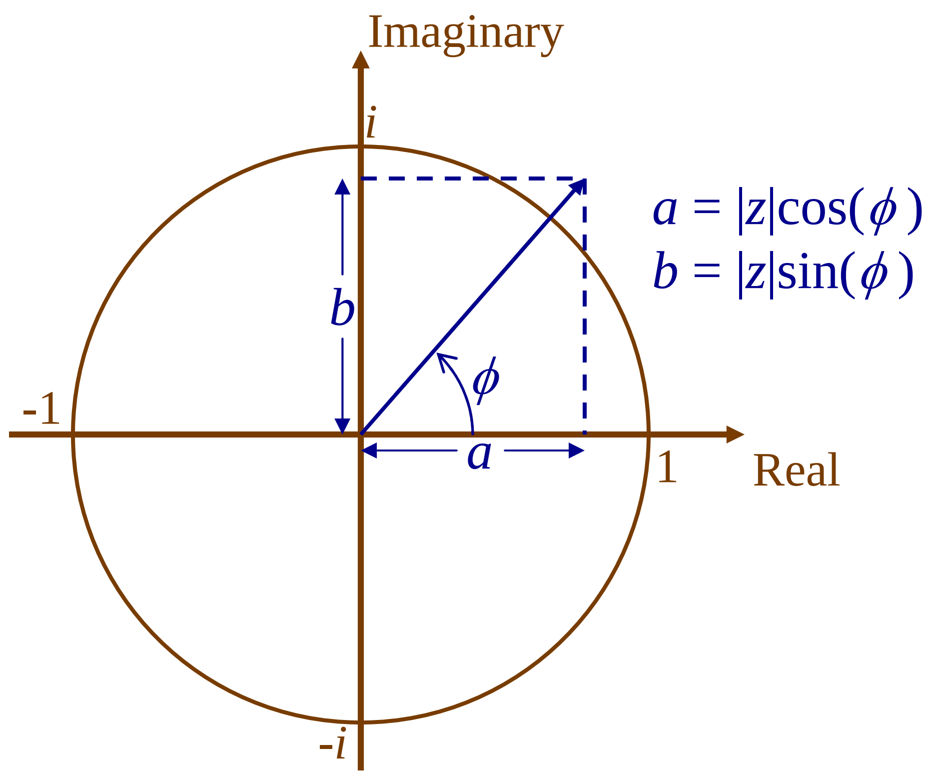
$$(m \cdot c^a)\,(n \cdot c^b) = m \cdot n \cdot c^{a+b}$$

- What constant c can represent complex numbers?

Euler's Formula

- Any complex number can be represented using Euler's formula:

$$z = |z|e^{i\phi(z)} = |z|\cos(\phi) + |z|\sin(\phi)i = a + bi$$



Powers of Complex Numbers

Suppose that we take a complex number

$$z = |z| e^{i\,\phi(z)}$$

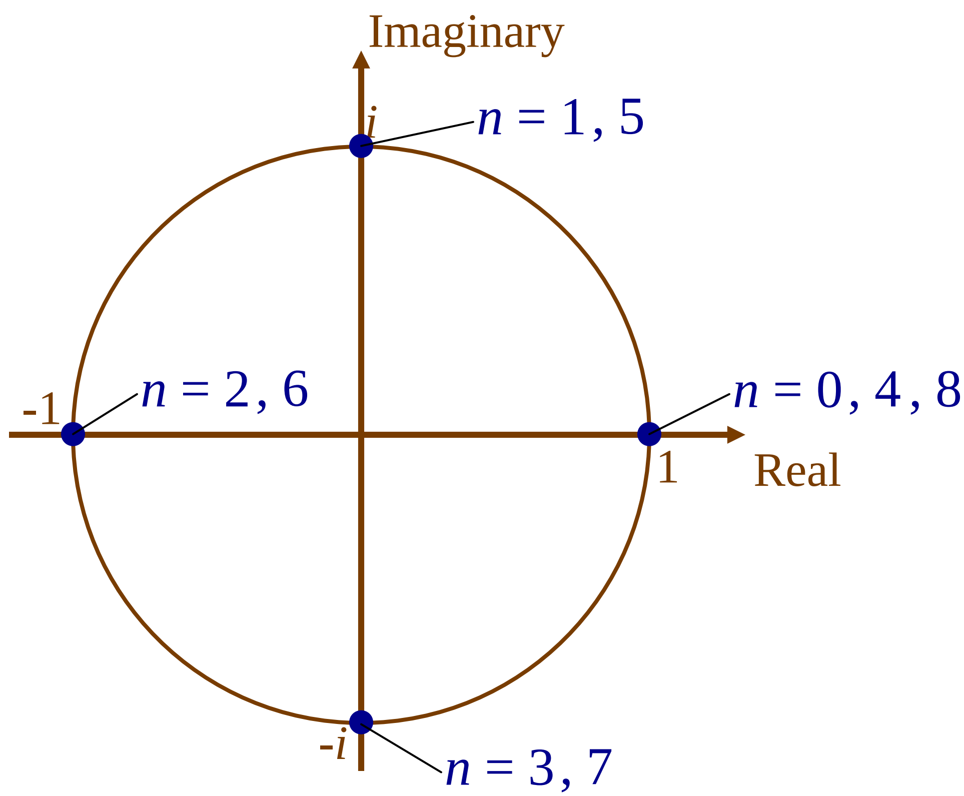
and raise it to some power

$$z^n = [|z| e^{i\,\phi(z)}]^n$$
$$= |z|^n \, e^{i\,n\,\phi(z)}$$

z^n has magnitude $|z|^n$ and phase $n\,\phi(z)$

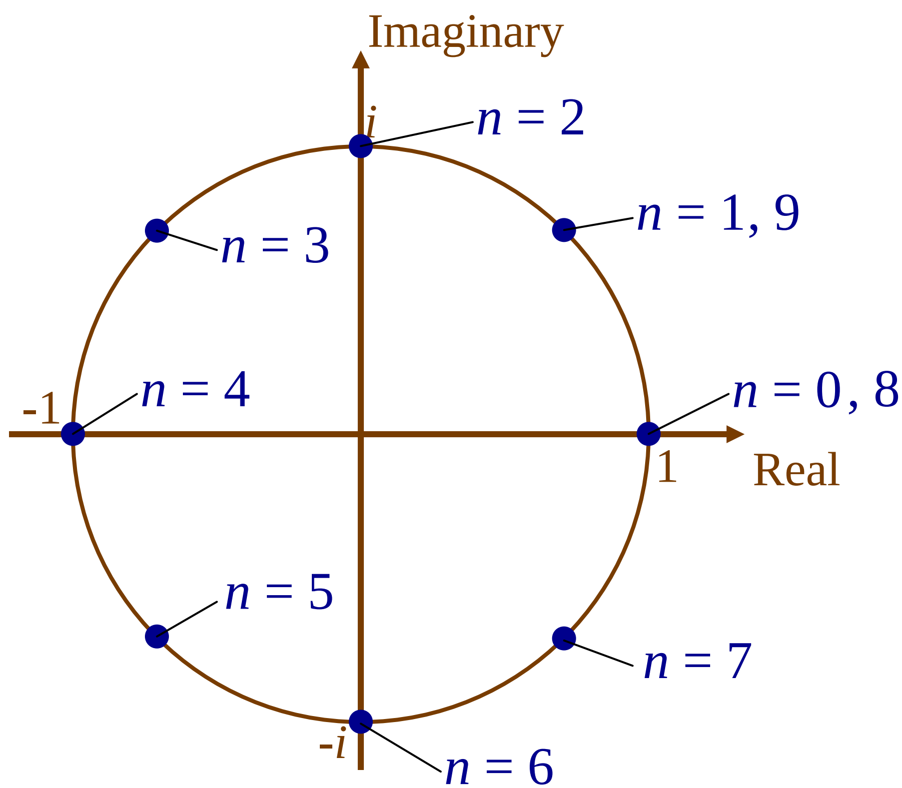
Powers of Complex Numbers: Example

- What is i^n for various n?



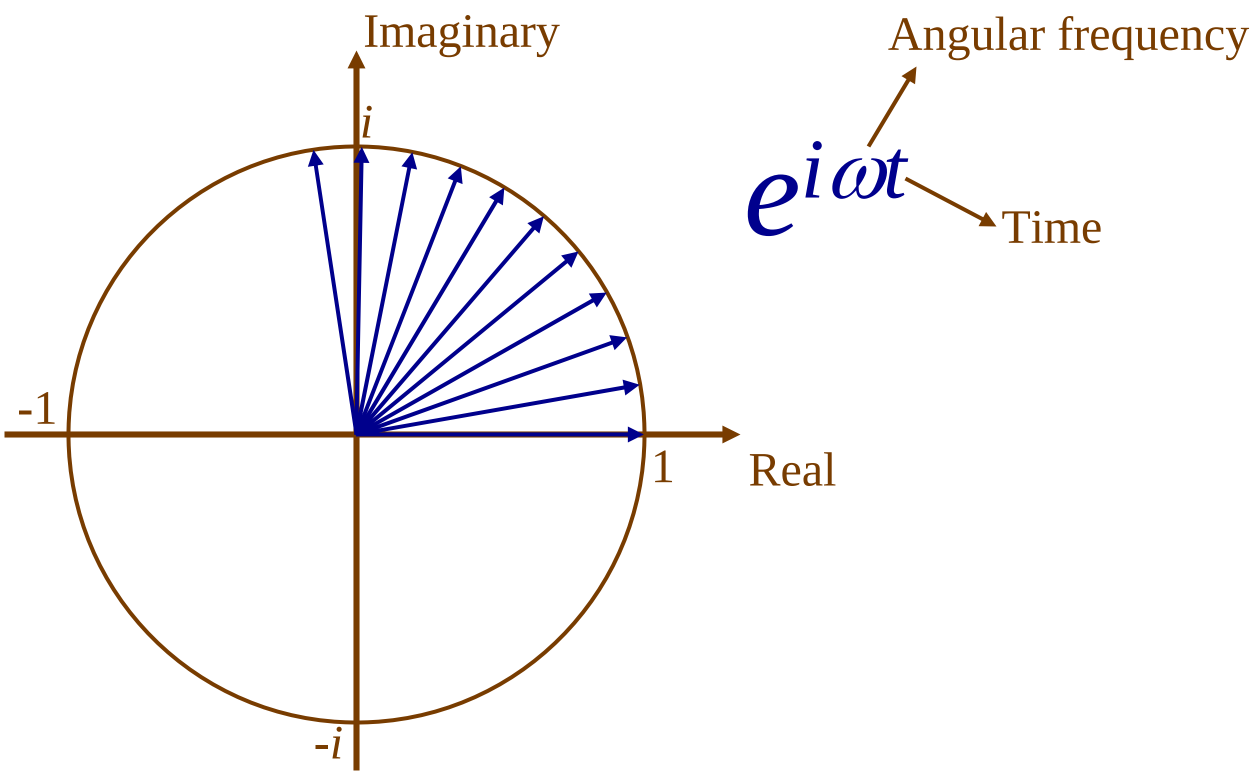
Powers of Complex Numbers: Example

- What is $(e^{i\pi/4})^n$ for various n?



Harmonic Functions

- What does $x(t) = e^{i\omega t}$ look like?
- $x(t)$ is a harmonic function (a building block for later analysis)



Harmonic Functions as Sinusoids

Real Part	Imaginary Part
$\Re(e^{i\omega t})$	$\Im(e^{i\omega t})$
$\cos(\omega t)$	$\sin(\omega t)$

Harmonics and Systems

If we present a harmonic input (function)

$$x_1(t) = e^{i\omega t}$$

to a shift-invariant linear system, it produces the response

$$x_1(t) \rightarrow y_1(t)$$

$$y_1(t) = K(\omega, t)\, x_1(t) = K(\omega, t)\, e^{i\omega t}$$

where, for now, we simply define

$$K(\omega, t) = \frac{y_1(t)}{e^{i\omega t}}$$

Harmonics and Systems: Shifted Input

Suppose we create a harmonic input (function) by shifting the original input

$$x_2(t) = x_1(t - T) = e^{i\omega(t - T)}$$

The response, $x_2(t) \rightarrow y_2(t)$, to this shifted input is

$$y_2(t) = K(\omega, t - T)\, x_2(t) = K(\omega, t - T)\, e^{i\omega(t - T)}$$

Harmonics and Systems: Shifted Input

However, note that

$$x_2(t) = e^{i\omega(t-T)} = e^{i\omega t}\, e^{-i\omega T} = x_1(t)\, e^{-i\omega T}$$

Thus, the response can be written

$$x_2(t) \rightarrow y_1(t)\, e^{-i\omega T}$$

$$y_2(t) = y_1(t)\, e^{-i\omega T} = K(\omega, t)\, x_1(t)\, e^{-i\omega T} =$$

$$K(\omega, t)\, x_2(t)$$

Harmonics and Systems: Shifted Input

Now we have both

$$y_2(t) = K(\omega, t)\, x_2(t)$$

$$y_2(t) = K(\omega, t - T)\, x_2(t)$$

Thus,

$$K(\omega, t - T) = K(\omega, t)$$

So, K is just a constant function with respect to t:

$$K(\omega)$$

Harmonics and Systems

Thus, for any harmonic function

$$x(t) = e^{i\omega t}$$

we have

$$x(t) \rightarrow y(t)$$

$$y(t) = K(\omega)\, x(t) = K(\omega)\, e^{i\omega t}$$

Implication: When a system (a shift-invariant linear transformation) is applied to a harmonic signal, the result is the same harmonic signal multiplied by a constant that depends only on the frequency

Transfer Functions

We now have a second way to characterize systems:

1: If you know the *impulse response* at any point, you know everything there is to know about the system

2: The function $K(\omega)$ defines the degree to which harmonic inputs transfer to the output

$K(\omega)$ is the called the *transfer function*

Transfer Functions

Expressing $K(\omega)$ in polar (magnitude-phase) form:

$$K(\omega) = A(\omega)\, e^{i\phi(\omega)}$$

Recall that the magnitudes multiply and the phases add:

$$K(\omega)\, e^{i\omega t} = A(\omega)\, e^{i[\omega t + \phi(\omega)]}$$

$A(\omega)$ is called the Modulation Transfer Function (MTF)

- Magnitude of the transfer function
- Indicates how much the system amplifies or attenuates input

$\phi(w)$ is called the Phase Transfer Function (PTF)

- Phase of the transfer function
- Only effect is to shift the time origin of the input function

Impulse Response

Remember that we can entirely characterize a system by its impulse response:

$$\delta(t) \rightarrow h(t)$$

Problem: given an input signal $x(t)$, how do we determine the output $y(t)$

Linearity and Shift Invariance

Because a system is linear:

$$a\,\delta(t) \rightarrow a\,h(t)$$

Because a system is shift invariant:

$$\delta(t-k) \rightarrow h(t-k)$$

Response to an Entire Signal

A signal $x(t)$ can be thought of as the sum of a lot of weighted, shifted impulse functions:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

where

- $\delta(\tau - t)$ is the delta function at τ
- $x(t)$ is the weight of that delta function

(Read the integral simply as summation)

Response to an Entire Signal (cont.)

Because of linearity, each impulse goes through the system separately:

$$x(\tau)\,\delta\,(t-\tau) \rightarrow x(\tau)\,h(t-\tau)$$

This means

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Response to an Entire Signal (cont.)

So,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

This operation is called the *convolution* of *x* and *h*

Convolution

Convolution of an input $x(t)$ with the impulse response $h(t)$ is written as

$$x(t) * h(t)$$

That is to say,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Response to an Entire Signal

So, the response of a system with impulse response $h(t)$ to input $x(t)$ is simply the convolution of $x(t)$ and $h(t)$:

$$x(t) \rightarrow y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

Convolution of Discrete Functions

For a discrete function $x[j]$ and impulse response $h[j]$:

$$x[j] * h[j] = \sum_{k} x[k] \cdot h[j-k]$$

One Way to Think of Convolution

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$x[j] * h[j] = \sum_{k} x[k] \cdot h[j-k]$$

Think of it this way:

- Shift a copy of *h* to each position *t* (or discrete position *k*)
- Multiply by the value at that position *x*(*t*) (or discrete sample *x*[*k*])
- Add shifted, multiplied copies for all *t* (or discrete *k*)

Example: Convolution – One way

$$x[j] = [\ 1\ \ 4\ \ 3\ \ 1\ \ 2\ ]$$

$$h[j] = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ ]$$

$$x[0]\, h[j-0] = [\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ ]$$

$$x[1]\, h[j-1] = [\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ ]$$

$$x[2]\, h[j-2] = [\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ ]$$

$$x[3]\, h[j-3] = [\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ ]$$

$$x[4]\, h[j-4] = [\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ ]$$

$$x[j] * h[j] = \sum_k x[k]\, h[j-k]$$

$$= [\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ \_\_\ ]$$

Example: Convolution – One way

$$x[j] = [\ 1\quad 4\quad 3\quad 1\quad 2\ ]$$

$$h[j] = [\ 1\quad 2\quad 3\quad 4\quad 5\ ]$$

$$x[0]\,h[j-0] = [\ 1\quad 2\quad 3\quad 4\quad 5\quad \_\_\quad \_\_\quad \_\_\quad \_\_\ ]$$

$$x[1]\,h[j-1] = [\ \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\ ]$$

$$x[2]\,h[j-2] = [\ \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\ ]$$

$$x[3]\,h[j-3] = [\ \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\ ]$$

$$x[4]\,h[j-4] = [\ \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\ ]$$

$$x[j] * h[j] = \sum_k x[k]\,h[j-k]$$

$$= [\ \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\quad \_\_\ ]$$

Example: Convolution – One way

$$x[j] = [\ 1 \quad 4 \quad 3 \quad 1 \quad 2\ ]$$

$$h[j] = [\ 1 \quad 2 \quad 3 \quad 4 \quad 5\ ]$$

$$x[0]\, h[j-0] = [\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[1]\, h[j-1] = [\ \_\_ \quad 4 \quad 8 \quad 12 \quad 16 \quad 20 \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[2]\, h[j-2] = [\ \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[3]\, h[j-3] = [\ \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[4]\, h[j-4] = [\ \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[j] * h[j] = \sum_k x[k]\, h[j-k]$$

$$= [\ \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

Example: Convolution – One way

$$x[j] = [\ 1\ \ 4\ \ 3\ \ 1\ \ 2\ ]$$
$$h[j] = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ ]$$

$$x[0]\,h[j-0] = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$
$$x[1]\,h[j-1] = [\ \_\_\ \ 4\ \ 8\ \ 12\ \ 16\ \ 20\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$
$$x[2]\,h[j-2] = [\ \_\_\ \ \_\_\ \ 3\ \ 6\ \ 9\ \ 12\ \ 15\ \ \_\_\ \ \_\_\ ]$$
$$x[3]\,h[j-3] = [\ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$
$$x[4]\,h[j-4] = [\ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

$$x[j] * h[j] = \sum_k x[k]\,h[j-k]$$
$$= [\ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

Example: Convolution – One way

$$x[j] = [\ 1 \quad 4 \quad 3 \quad 1 \quad 2\ ]$$

$$h[j] = [\ 1 \quad 2 \quad 3 \quad 4 \quad 5\ ]$$

$$x[0]\,h[j-0] = [\ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[1]\,h[j-1] = [\ \_\_ \quad 4 \quad 8 \quad 12 \quad 16 \quad 20 \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[2]\,h[j-2] = [\ \_\_ \quad \_\_ \quad 3 \quad 6 \quad 9 \quad 12 \quad 15 \quad \_\_ \quad \_\_\ ]$$

$$x[3]\,h[j-3] = [\ \_\_ \quad \_\_ \quad \_\_ \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad \_\_\ ]$$

$$x[4]\,h[j-4] = [\ \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

$$x[j] * h[j] = \sum_k x[k]\,h[j-k]$$

$$= [\ \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_ \quad \_\_\ ]$$

Example: Convolution – One way

$$x[j] = [\ 1\ \ 4\ \ 3\ \ 1\ \ 2\ ]$$

$$h[j] = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ ]$$

$$x[0]\, h[j-0] = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

$$x[1]\, h[j-1] = [\ \_\_\ \ 4\ \ 8\ \ 12\ \ 16\ \ 20\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

$$x[2]\, h[j-2] = [\ \_\_\ \ \_\_\ \ 3\ \ 6\ \ 9\ \ 12\ \ 15\ \ \_\_\ \ \_\_\ ]$$

$$x[3]\, h[j-3] = [\ \_\_\ \ \_\_\ \ \_\_\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \_\_\ ]$$

$$x[4]\, h[j-4] = [\ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ 2\ \ 4\ \ 6\ \ 8\ \ 10\ ]$$

$$x[j] * h[j] = \sum_k x[k]\, h[j-k]$$

$$= [\ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

Example: Convolution – One way

$$x[j] = [\ 1\ \ 4\ \ 3\ \ 1\ \ 2\ ]$$

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$$x[0]\,h[j-0] = [\ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

$$x[1]\,h[j-1] = [\ \_\_\ \ 4\ \ 8\ \ 12\ \ 16\ \ 20\ \ \_\_\ \ \_\_\ \ \_\_\ ]$$

$$x[2]\,h[j-2] = [\ \_\_\ \ \_\_\ \ 3\ \ 6\ \ 9\ \ 12\ \ 15\ \ \_\_\ \ \_\_\ ]$$

$$x[3]\,h[j-3] = [\ \_\_\ \ \_\_\ \ \_\_\ \ 1\ \ 2\ \ 3\ \ 4\ \ 5\ \ \_\_\ ]$$

$$x[4]\,h[j-4] = [\ \_\_\ \ \_\_\ \ \_\_\ \ \_\_\ \ 2\ \ 4\ \ 6\ \ 8\ \ 10\ ]$$

$$x[j] * h[j] = \sum_k x[k]\,h[j-k]$$

$$= [\ 1\ \ 6\ \ 14\ \ 23\ \ 34\ \ 39\ \ 25\ \ 13\ \ 10\ ]$$

Another Way to Look at Convolution

$$x[j] * h[j] = \sum_k x[k] \cdot h[j-k]$$

Think of it this way:

– Flip the function *h* around zero

– Shift a copy to output position *j*

– Point-wise multiply for each position *k* the value of the function *x* and the flipped and shifted copy of *h*

– Add for all *k* and write that value at position *j*

Why Flip the Impulse Function?

An input at t produces a response at $t + \tau$ of $h(\tau)$

Suppose I want to determine the output at t

What effect does the input at $t + \tau$ have on t?

$$h(-\tau)$$

Convolution in Two Dimensions

In one dimension:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

In two dimensions:

$$I(x,y) * h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(\tau_x, \tau_y) h(x-\tau_x, y-\tau_y) d\tau_x d\tau_y$$

Or, in discrete form:

$$I[x,y] * h[x,y] = \sum_k \sum_j I[j,k] h[x-j, y-k]$$

Example: Two-Dimensional Convolution

$$\begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{matrix} \quad * \quad \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{matrix} \quad =$$

______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______
______	______	______	______	______	______

Example: Two-Dimensional Convolution

$$
\begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{matrix}
\quad * \quad
\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{matrix}
\quad =
$$

$$
\begin{matrix} 1 & 2 & 4 & 5 & 4 & 2 \\ 2 & 5 & 9 & 12 & 10 & 4 \\ 3 & 7 & 13 & 17 & 14 & 6 \\ 3 & 7 & 13 & 17 & 14 & 6 \\ 2 & 5 & 9 & 12 & 10 & 4 \\ 1 & 2 & 4 & 5 & 4 & 2 \end{matrix}
$$

Properties of Convolution

- Commutative: $f * g = g * f$
- Associative: $f * (g * h) = (f * g) * h$
- Distributive over addition: $f * (g + h) = f * g + f * h$
- Derivative: $$\frac{d}{dt}(f * g) = f' * g + f * g'$$

Convolution has the same mathematical properties as multiplication

(This is no coincidence)

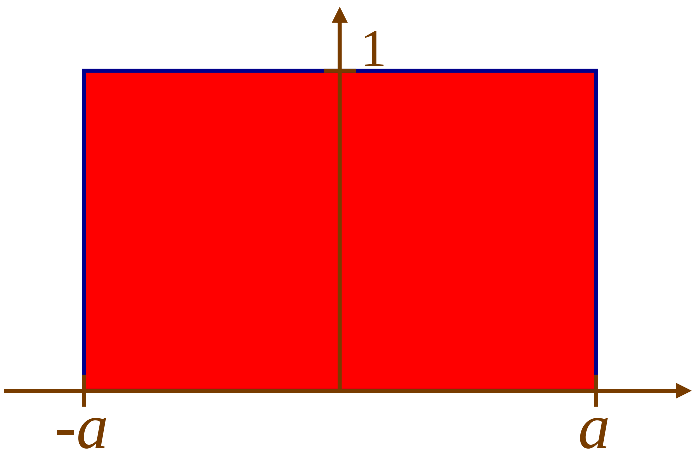
Useful Functions

- Square: $\Pi_a(t)$
- Triangle: $\Lambda_a(t)$
- Gaussian: $G(t, s)$
- Step: $u(t)$
- Impulse/Delta: $\delta(t)$
- Comb (Shah Function): $\text{comb}_h(t)$

Each has their two-dimensional equivalent.

Square

$$\Pi_a(t) = \begin{cases} 1 & \text{if } |t| \le a \\ 0 & \text{otherwise} \end{cases}$$

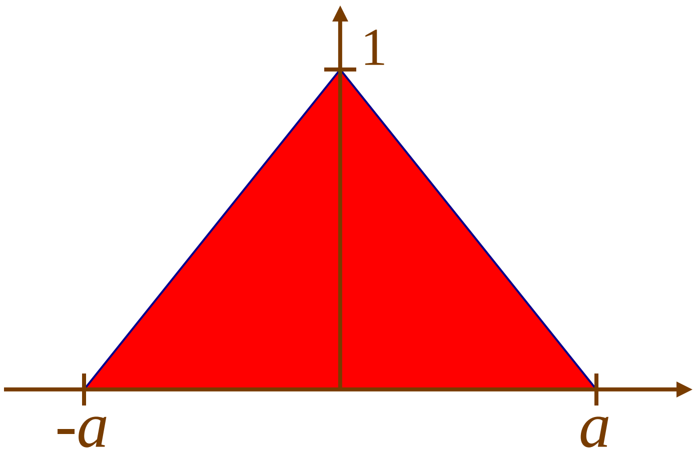


What does $f(t) * \Pi_a(t)$ do to a signal $f(t)$?

What is $\Pi_a(t) * \Pi_a(t)$?

Triangle

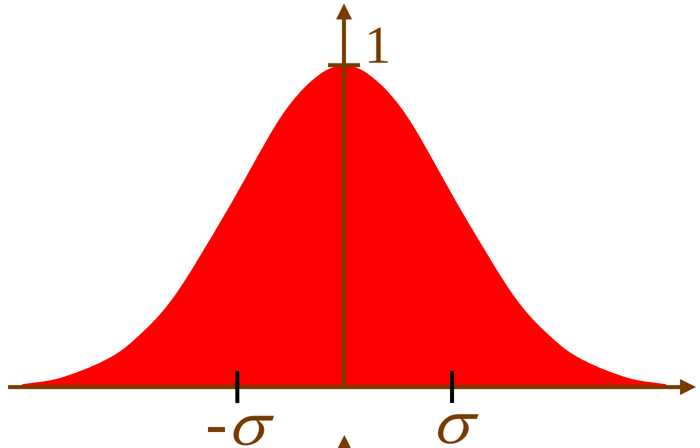
$$\Lambda_a(t) = \begin{cases} 1 - \left| t/a \right| & \text{if } |t| \le a \\ 0 & \text{otherwise} \end{cases}$$



Gaussian

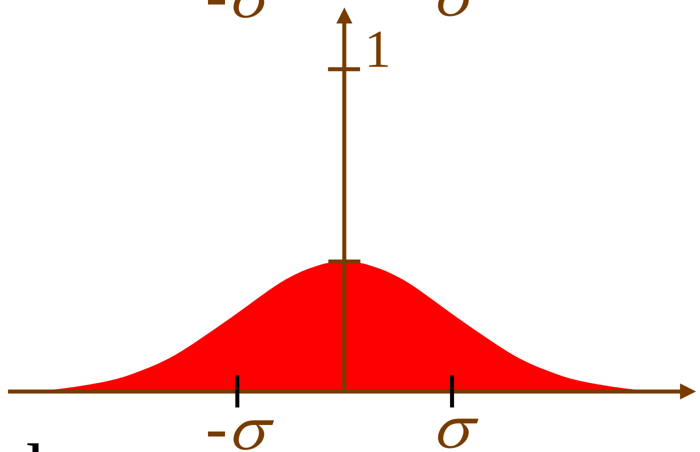
Gaussian: maximum value = 1

$$G(t,\sigma) = e^{-t^2 / 2\sigma^2}$$



Normalized Gaussian: area = 1

$$G(t,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-t^2 / 2\sigma^2}$$

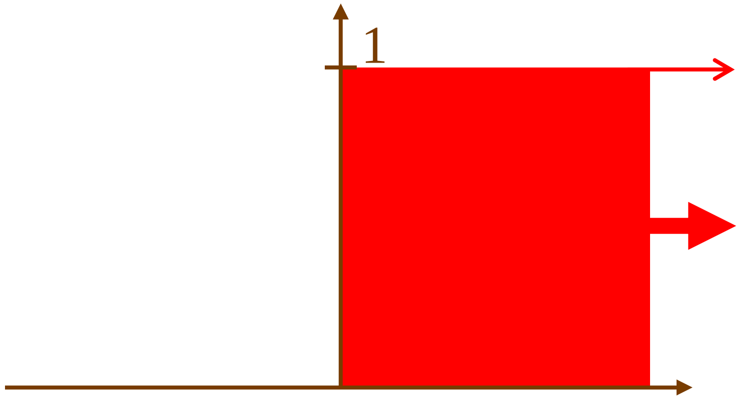


Convolving a Gaussian with another:

$$G(t,\sigma_1) * G(t,\sigma_2) = G(t, \sqrt{\sigma_1^2 + \sigma_2^2})$$

Step Function

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

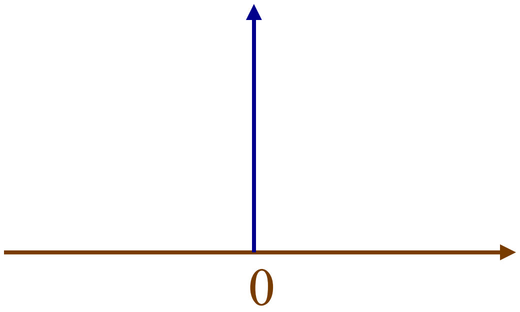


What is the derivative of a step function?

Impulse/Delta Function

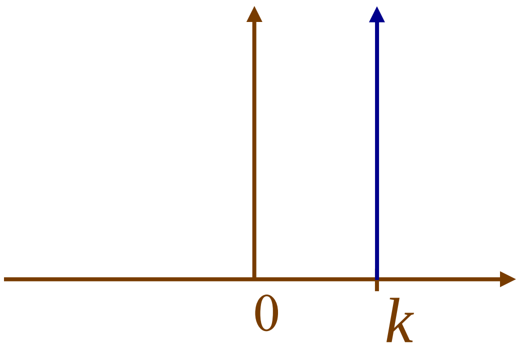
- We’ve seen the delta function before:

$$\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$



- Shifted Delta function: impulse at t = k

$$\delta(t-k) = \begin{cases} \infty & \text{if } t = k \\ 0 & \text{otherwise} \end{cases}$$

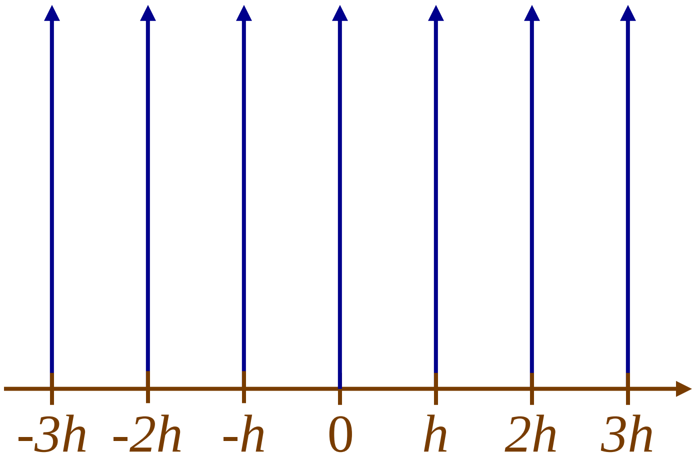


- What is a function $f(t)$ convolved with $\delta(t)$?
- What is a function $f(t)$ convolved with $\delta(t - k)$?

Comb (Shah) Function

A set of equally-spaced impulses: also called an impulse train

$$comb_h(t) = \sum_k \delta(t - hk)$$



h is the spacing

What is $f(t) * \text{comb}_h(t)$?

Convolution Filtering

- Convolution is useful for modeling the behavior of linear, shift-invariant devices
- It is also useful to do ourselves to produce a desired effect
- When we do it ourselves, we get to choose the function that the input will be convolved with
- This function that is convolved with the input is called the *convolution kernel*

Convolution Filtering: Averaging

Can use a square function ("box filter") or Gaussian to locally average the signal/image

– Square (box) function: uniform averaging

– Gaussian: center-weighted averaging

Both of these blur the signal or image

Convolution Filtering: Unsharp Masking

To sharpen a signal/image, subtract a little bit of the blurred input:

$$I(x, y) + \alpha\,[I(x, y) - I(x, y) * G(x, y, \sigma)]$$

This is called unsharp masking

The larger you make α, the more sharpening you get

More on filters in later sessions!

Text Credits

The text for this lecture was adapted in part from the following source, in agreement with the listed copyright statements:

http://web.engr.oregonstate.edu/~enm/cs519/index.html

Resources

Textbooks:
Kenneth R. Castleman, Digital Image Processing, Chapter 9